THE

MATHEMATICAL

GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

62 Coleraine Road, Blackheath, London, S.E. 3.

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VOL. XXIII. JULY, 1939. No. 255

DIVISION BY ITERATION.

BY EDWARD B. ESCOTT.

THIS method of division is related to several methods which have been published from time to time.* The advantage of the method given below is the advantage that iteration has, of substituting a repetition of a number of easy operations for a more difficult one, and the fact that at each operation the work starts at the beginning so that errors (except in the last operation) are automatically corrected.

Division by a number ending in zeroes is easily done by cancelling the zeroes in the divisor and moving the decimal point in the dividend the same number of places to the left as the number of zeroes cancelled. Then divide by the shortened divisor.

Division by numbers ending in 1 *or* 9 may be shortened by reducing the problem to the preceding. This will be shown by examples.

Division by 9. The dividend is 9 times the quotient. If I add the quotient to the dividend, the sum is 10 times the quotient. That is, the sum will be the same as the quotient written one place to the left.

Starting with the first figure of the quotient (obtained by inspection), the sum of the dividend and the quotient figure will be the quotient to two figures. Correct the quotient by annexing the second figure just found and repeat the operation. This will give the quotient to three figures. Repeat the operation until the quotient is found to the desired number of figures.

* A. A. Fletcher-Jones, " Method of Division for Small Divisors ", *Mathematical Gazette*, Vol. 20 (Dec. 1936), 331-2. Notes on the above paper, *Mathematical Gazette*, Vol. 21 (Oct. 1937), 289-292, by J. C., C. D. Langford, and E. B. Escott. In the latter are given a number of references. On page 291, line 5, " Suffolk " should be corrected to " Suffield ".

R

Example 1. Divide 340672189273 by 9.

```
9 ) 340672189273
    3
    37
```

The first figure of the quotient 3 is found by inspection. (If through carelessness the figure is taken too large or too small, the correct value will appear at once.)

Adding the assumed quotient 3 to the dividend, the sum 37 is the quotient to two figures. Correct the quotient by annexing the second figure 7 and adding to the dividend, giving 377. Annex the third figure 7 to the quotient and adding, the sum 3783 shows that the third figure should be 8. Making the correction and continuing the work, the quotient is 37852465474 which was obtained without actual division.

Inasmuch as the figures of the quotient have been corrected many times, the safest way to be sure of the correctness of the quotient is to write in a new place on the paper the original problem, and below the dividend the quotient just found. Add the dividend and quotient. The sum should be 10 times the quotient. This is the case, proving the correctness of the work.

```
9 ) 340672189273
     37852465474
    378524654747
```

The same method exactly will be used in dividing by 99, by 999, by 9999, etc.

Example 2. Divide 321795062297 by 99.

```
99 ) 321795062297
       3250455174
     325045517471
```

Division by 19. Since the dividend is 19 times the quotient, the sum of dividend and quotient is 20 times the quotient. Dividing this sum by 2, the result is 10 times the quotient, giving the quotient to one more figure. Correct the quotient and repeat the work.

Example 3. Divide 216890237568 by 19.

```
19 ) 216890237568
      11415275661
 2 ) 228305513229
     114152756614
```

The first figure 1 of the quotient is found by inspection. Adding this to the dividend and dividing the sum by 2, I have 11, the quotient to two figures. Correcting the assumed quotient by annexing 1 and repeating the work I find the next value of the quotient 114. Annex the third figure to the quotient and continue the work as far as desired.

As shown, the result checks itself.

By the same method we can divide by 199, by 1999, by 19999, etc. Also division by 29, 299, 2999, ... is similar; also division by any one of the nine digits followed by any number of 9's.

Division by numbers of two figures ending in 1. The method is similar to the preceding, except that the quotient is subtracted from the dividend, instead of being added to it.

Division by 11. Since the dividend is 11 times the quotient, the dividend minus the quotient is equal to 10 times the quotient or is the quotient written one place to the left.

EXAMPLE 4. Divide 302674518824 by 11.

```
11 ) 302674518824
      27515865347
     275158653477
```

The first figure of the quotient (found by inspection) 2. Subtracting this quotient figure from the dividend the difference 28 is the quotient to two figures. Annex the second figure 8 to the quotient figure 2 and subtract, giving for the corrected quotient 274, etc.

The same method enables us to divide by 101, 1001, 10001, etc.

Division by 71.

EXAMPLE 5. Divide 42470231765 by 71.

```
71 ) 42470231765
       598172278
 7 ) 41872059487
      5981722783
```

Since the dividend is 71 times the quotient, the dividend minus the quotient is 70 times the quotient. Dividing this difference by 7 gives 10 times the quotient. Annex the second figure to the quotient, etc. The complete quotient is 598172278.

By the same method we can divide by 701, 7001, 70001, etc. Division by any of the nine digits followed by 1 is similar.

Division by a number ending in 3 *or* 7. Multiply both dividend and divisor by a number which will make the new divisor end in 1 or 9. Thus, to divide by 13, multiply divisor and dividend by 3 or 7. To divide by 17, multiply divisor and dividend by 3 or 7.*

EXAMPLE 6. Divide 316420705523 by 17. Multiply both divisor and dividend by 3, making the problem to divide 949262116569 by 51.

```
51 ) 949262116569
       1
 5 ) 93
      18
```

By inspection the first quotient figure is 1. Since the dividend is 51 times the quotient, the dividend minus the quotient is 50 times the quotient. This difference divided by 5 gives 10 times the quotient, 18, etc.

```
51 ) 949262116569
       1
      93
    × 2 = 186
```

Instead of dividing the difference by 5, a better way would be to multiply it by 2, giving 100 times the quotient, as follows :

Other numbers may also be used as divisors. There are other numbers by which division can be performed by short division. Some of these numbers are 12, 15, 125, 1125. For instance, to

* For a list of multipliers suitable for reducing many divisors to suitable form see page 291 of my note. (See foot-note, page 241, of this paper.)

divide by 12: divide divisor and dividend by 6, making the new divisor 2. The difference of these two divisors is 10.

EXAMPLE 7.	12) 3125097846
Divide by 6	2) 520849641
Subtract	10) 2604248205

From this we can easily divide by 119 or 121, as below:

EXAMPLE 8.	119) 214067498327
	1798886540
	12) 215866384867
	2) 35977730811
Subtract	10) 179888654056

In the same way, we can divide by 1199 11999, ..., and in a similar way, we can divide by 121, 1201, 12001, ...

To divide by 15. Divide divisor and dividend by 3, making the new divisor 5. The difference of these two divisors is 10.

EXAMPLE 9.	149) 35417289453
	2
	15) 356
Divide by 3	5) 118
	10) 238

The first figure of the quotient is 2. Add this quotient figure to the dividend, making the sum 150 times the quotient. Divide the new divisor and dividend by 3, and subtract. Then the first three figures of the quotient are 238. Repeat as many times as necessary.

In the same way, we may divide by 1499, 14999, ... In a similar way, we can divide by 151, 1501, 15001, ...

To divide by 125. One way is to multiply divisor and dividend by 8, making the new divisor 1000. A simpler way is to multiply divisor and dividend by 2, writing the products one place to the right, then subtract.

EXAMPLE 10. Divide	125) 68743298077103
Multiply by 2	250) 137486596154206
Subtract	1000) 549946384616824

EXAMPLE 11. Divide	1249) 5438732902189
	4
	125) 5442
Multiply by 2	250) 10884
Subtract	1000) 43536

In the same way, we can divide by 12499, 124999, ... Similarly, we can divide by 126, 1251, 12501, ...

To divide by 1125. Divide divisor and dividend by 9, making the new divisor 125. Then take the differences of the divisors for a new divisor 1000.

EXAMPLE 12. Divide 11249) 416743897065421
3
1125) 41677
Divide by 9 125) 4630
Subtract 1000) 37047

Other numbers than those mentioned which can be used for divisors are 75 and 875. To divide by 75, divide divisor and dividend by 3, making the new divisor 25. Add the last two divisors, giving 100.

EXAMPLE 13. Divide 74) 416287905432764
5
75) 421
Divide by 3 25) 140
Add 100) 561

In the same way, we can divide by 749, 7499, ... , and in a similar way by 76, 751, 7501, 75001, ...

To divide by 875. Divide divisor and dividend by 7, making the new divisor 125. The sum of these two divisors is 1000.

EXAMPLE 14. Divide 874) 210965532871
241379328
875) 211206912199
Divide by 7 125) 30172416028
Add 1000) 241379328227

In the same way we can divide by 8749, 87499, ... ; and in a similar way we can divide by 876, 8751, 87501, 875001, ...

The method may be applied to division by numbers which do not come under the preceding cases, especially to numbers most of whose digits are either all small or all large. Here are some examples:

49989) 612036527931
1
1
6121

It is necessary to add 11 times the quotient to the dividend. Then multiply this sum by 2, giving the quotient to six figures. Correct the quotient by annexing one or two figures and repeat operation. The completed work is shown below.

49989) 612036527931
12243424
122434241
612171205596
$\times 2 =$ 1224342411192

30021) 742103865782
24719491
494389837
3) 741584756454
247194918818

Subtract 21 times the quotient from the dividend by the method of subtracting the sum of the second and third lines from the first line.

Divide this difference by 3, giving the quotient to five figures.

E. B. E.

SCHOOL ALGEBRA OF THE FIRST YEAR.

By G. Lawson.

As his presidential lecture the President of the Edinburgh Mathematical Society read, on November 4, 1938, the opening paper of a debate on School Algebra of the First Year. The algebra of the Mathematical Association 1933 Report and the 1937 British Association Colloquium was for short referred to as 1933-1937 algebra or as N-D-S algebra (Nunn-Durell-Siddons), and was taken as an authoritative map of the present position in first year algebra.

Assuming to be hopeless any debate between the older condemned algebra and the N-D-S algebra, the President found an only hope of debate in criticism of the 1933-1937 map. As his own qualification for offering criticism he put forward that he (1) had taught mathematics for over forty years, (2) had, with the assistance of two members of staff, made a long experiment in algebra of beginners. He could therefore claim to cast a fairly knowing eye over the 1933-1937 map. This eye descried in the map three distinct, though interconnected, more or less blighted areas.

(A) Looking back over experiments, said the President, I can see myself driven from one position to another, by failures and dissatisfactions, back and back, not to Durell formulae, not to Siddons equations, but past these, which are after all only applications of algebra, back to what I have come to regard as the fundamental thing in the teaching of algebra, namely, Form.

In the 1933-1937 algebra the idea of form simply has not arrived . . . if indeed it is not, at page 8 of 1933, deliberately excluded from a beginner's course ; in the voluminous index of the Nunn classic the word form does not occur!

So much for the first area of poor nutrition, (A) ; from it my eye travels to the second, (B), where in place of an adequate teaching of law is to be found a sorry substitute in the shape of rules for removing brackets, a substitute which from my viewpoint is almost a necessary consequence of the earlier ignorance or ignoration of form.

At this stage I interject . . . please note for later reference . . . that an early decision in my experimentation was, that the first term teaching which is most practically effective, seeming rather slow at the start of the term but paying best by the end, is teaching, not indeed of theory, but teaching, with skill, *along lines of good theory*. For instance, Chrystal makes some early theoretical play with the " startling " " duality " of algebra : but he never suspected, never had occasion to suspect, that a teacher in search of good practice would find this theoretic duality of his an effective tool in the teaching of beginners. This tool is unknown in 1933-1937, though at page 108 of 1933 I can see a hand fumbling, as once my own fumbled, to get hold of the tool.

(C) The third blighted area in the 1933-1937 map is the Negative

Number; and now, *constructive* criticism; make yourselves a class of beginners to whom I am giving your first lesson on negatives, but at the same time be experts alert to criticise and debate.

In the early days there were no fractions in numberland, only the natural numbers 1, 2, 3, etc., the earliest settlers in the land. *They* invented the forms of addition, etc.; *they* made and handed down the laws of add-subtract forms and of multiply-divide forms; every law of the land, *without exception*, dates back to the ancient times of these first settlers.

In these early days a form like 12 divided by 4 or 12 over 4 was allowed, because there is a natural number which can be the tally of the form. But a multiply-divide form like 15 over 7, or 3 over 5, was rejected as impossible, because there is no natural number which can be the tally of such form.

Rejected, at first; but later, for the sake of getting work done which they could not do themselves but which they much wished done, the old aristocrats decided to import slaves to do the work: slaves who came to be known as fractions while the ruling class styled themselves integers. That importation of slave fractions took place long ago, and now we are so used to meeting these slaves at every turn of arithmetical business that it is not easy for us to understand how old numberland ever got along without them.

But the study of ancient history is of little *practical* use unless it helps us to solve present-day problems. Think: in our course so far have we encountered any difficulty like that of the old settlers? There was work *they* could not do: is there any job we have found *we* cannot do? *They* rejected the form 15 over 7 as impossible: is there any form *we* have been rejecting as absurd and impossible? Think back. (Of course the absurdity of the add-subtract form $3-5$ has been harped on throughout.)

Follows a scale study: On blackboard make 12 notches of a vertical scale numbered from 0 to 11; apply it to "scale"

$$0+9-2+1-4+7,$$

placing pointer at 0, moving up 9 jumps, then down 2 jumps, etc., etc., arriving at 11.

Commute to $0+7+1-2-4+9$, and to $0+7-2-4+1+9$, and to $0+9-4-2+7+1$, and to $0+1+7-2-4+9$ (all carefully chosen so that all scale-motion is inside the range 0 to 11). Always arriving at 11.

Next commute to $0+7-2+9-4+1$, and try to scale: *failure*; the scale is not high enough; easily remedied; extend notches up to 14, but *do not write* the names above 11; merely make the notches, and write "Unnamed Numbers" at these notches. Next commute to $0+9+7+1-2-4$, requiring three more notches among unnamed numbers at the top. Every final is 11.

Commute to $0+1-4+7-2+9$: try to scale: *fail*; but we now know what to do; make notches below, just as we need them. A boy of mine objected to this notching below—because "*there are*

no names left," he said. This is an objection worth provoking : and meeting ; no new notches because we have not got names for them? no flying machines because we have not got the name aeroplane? Out-arguing the objection extend the notches down among "nameless numbers" : and find that with every possible commutation the final answer is 11.

Exercises on nameless numbers, *e.g.*, scale $0+4-9+6-13$ and mark with a star the final nameless answer ; commute, and every time arrive at the same starred nameless final.

Just as once upon a time numberland imported slaves to do work which the natural numbers could not do for themselves (describe this work again), so now, faced by work like $3-5$ which they cannot do for themselves, the old aristocrats import helots to do it for them. These helots (or robots?) are the nameless numbers on the scale.

Nameless as they are they do the job efficiently, as we have seen. But after all, we must get names for them : for we must order them about and write about them, not merely point to their notches on the scale.

Go back to history. How did the old settlers find names for the slave fractions? Simply enough ; when a slave landed he was branded with a double-barrelled name showing his exact business ; names like $\frac{15}{7}$ and $\frac{3}{8}$: and so with the new immigrants ; when a helot lands he is branded with a double-barrelled name showing his job, such as $3-5$ or $13-91$. The helots get the class name of negative numbers, and in contrast the old aristocrats are positive numbers.

Ex. By actual placing on the scale show $3-7$ greater than, *i.e.* higher than, $5-11$; arrange in descending order $5-12$, 2, and $4-9$.

Ex. The fraction $\frac{35}{56}$ has many different names, $\frac{5}{8}$, $\frac{15}{24}$, etc ; make up a few more of its double-barrelled names, and give the rule for manufacturing them. Show by actual placing on the scale that $5-11$, $3-9$, $23-29$ are all different names for the same helot negative number ; and give a rule for manufacturing different double-barrelled names of the same helot negative.

Ex. Of all the names, $\frac{35}{56}$, etc., for the same slave fraction, we say $\frac{5}{8}$ is the *simplest*, though it is still a double-barrelled name ; of the names $5-11$, $3-9$, $23-29$, etc., which do you fancy as the simplest?

Ex. Place the helot negative $7-15$ on the scale, and complete the following double-barrelled names for it : $6-?$, $5-?$, $4-?$, $3-?$, $2-?$, $1-?$, $0-?$. Of all these, the double-barrelled name beginning with Christian name 0 is considered to be the simplest. Make a scale, and attach to the negative numbers on it their simplest double-barrelled names.

At school Andrew Thomson and James Harvey are probably known as Thomson and Harvey, Christian names being dropped ; similarly the (bad?) practice has grown up of speaking and writing

about the negative number whose simplest proper name is $0-3$ merely as -3, the Christian name 0 being dropped.

A note for experts: Barnard and Child, working with a horizontal scale, " extend the scale to the left of zero by inventing the symbols -1, -2, etc. and placing -1 before 0, -2 before -1, etc.". Compare with this my boy's objection to the new notches because " there are no names left ".

And again: B. and C. hasten to say that the sign $-$ as used here has nothing whatever to do with subtraction. Tell that to a youngster who has gone through the preceding drill; he will ask you whether in the slave name $\frac{3}{5}$ there is no trace of division! (And yet, what about Jim Farmer? Is there anything about farming in his proper name?)

Passing on: The old slave immigrants are extremely useful members of the community; of what use are the new helots?

An exercise on permanence of form:

If Tom is 19 and Alec is 7, then Tom was born $19-7$ or 12 years before Alec.

If Tom is 21 and Alec 18, then Tom was born $21-18$ or 3 years before Alec.

If Tom is x and Alec is y, then Tom was born $x-y$ years before Alec.

If Tom is m and Alec is n, then Tom was born $m-n$ years before Alec.

If Tom is 3 and Alec is 5, then Tom was born $3-5$ or -2 years before Alec,

from which we see that to say Tom was born (-2) years before Alec really means etc.

In work with numbers hardly anything is better known than the laws of fractions—the ways of adding, subtracting, multiplying, and dividing fractions; we must learn how to add, subtract, multiply, and divide negative numbers.

First, as experts, not as beginners, observe the admittedly long and difficult discussions of these problems, in 1933-1937 and in one textbook after another of the N-D-S régime. (Some details were given, and quotation from W. J. D.'s review, Dec. 1937, of Durell's *School Certificate Algebra*.)

Our plan is to go back to history. How were the laws of fractions got? Go back to the first paragraph statement that all laws in numberland, without exception, date back to the ancient times when the only people in the land were the first settlers 1, 2, 3, etc. Absurd it seems to say that our familiar laws of fractions date back to a time before fractions were ever heard of; absurd though it sounds, true it is. For the multiply-divide forms $\frac{20}{5}$, $\frac{18}{3}$ etc. were in the land from the first, and the laws of these multiply-divide forms were known long before fractions were imported. When a fraction landed, he was first branded; but he was not turned loose to do as he liked and perhaps wreck the community: no, since he

belonged to the tribe of multiply-divide forms, he was told he must learn and obey the ancient laws of these forms: and thus it comes about that a fraction must make not a single move, by way of adding, subtracting, multiplying, or dividing, except in strict obedience to one or other of these ancient laws. So used are we to fractions obeying these laws that we have come to think and speak about the laws as the *laws of fractions*, as if they were labour laws specially made for fractions instead of being, as they actually are, the ancient laws of multiply-divide forms.

And now we know what to do. When a helot negative lands, he is branded with a double-barrelled name. He is not turned loose to lead a lawless life: he is told he belongs to the tribe of add-subtract forms and must learn and obey the ancient laws of these forms, and make not a single move except in accord with one or other of these ancient laws.

Thus $5+(-3)-(-7)$ becomes, when everybody gets his rightful name, $5+(0-3)-(0-7)$, which by ancient laws is $5+0-3-0+7$, etc. Again, $(-3)(-2)$ is, correctly and fully written, $(0-3)(0-2)$, which by ancient law becomes $0 \,.\, 0-0 \,.\, 2-3 \,.\, 0+3 \,.\, 2$, which = etc.

As to division, the usual way is the best; since $(-2)(-3)$ equals 6, therefore $6/(-2)$ must equal (-3).

To the expert this last has an interest. It seems to be a special labour law. When helot (-3) or $(0-3)$ wants to divide into 6, he must look about for an ancient law of his tribe, a law for $a/(b-c)$ or $60/(5-3)$. The expected law might be $a/b-a/c$ and $60/5-60/3$. But no such law among the ancient laws does the distressed $(0-3)$ find; his difficulty is solved by the special labour law.

A final remark. It was in the course of experimental investigation that these notes on negatives were planned, nearly ten years ago, more crudely than now set out; but it was only this year that one day, after briefly explaining the plan to a young teacher who had made trial of earlier notes on form, I suddenly realised that these practical lessons on negatives are an application to negatives of the theory of number pairs. And it is curious to note, that while the 1933 Report, page 96, discusses the fraction as a number pair, and at page 335 of the 1937 paper Mr. Langford pleads for sixth form tackle of the complex number as an ordered pair, alongside of this, in both papers, is set out at length the usual N-D-S treatment of negatives. It never seems to have crossed the mind of anyone concerned with *school* algebra that the negative number is also a number pair. And yet, turn to Hobson's *Theory of Functions of a Real Variable* and find there the complete theory; which makes me claim again that the most effective teaching is *along lines of good theory.* G. L.

GLEANINGS FAR AND NEAR.

1272. A cube is a solid square.—M. Armstrong, *The Paintbox*. [Per Mr. W. H. Beverstock.]

THE TEACHING OF "APPLIED" MATHEMATICS.*

BY A. BUXTON.

I.

ATTENTION has been recently drawn to the rapid mathematisation of science during the past half-century and the contributions to educational technique which are being made by teachers of mathematics in technical colleges.†

The kind of mathematics with which they deal may be called "Applied" Mathematics, although it is far different from what is ordinarily termed applied mathematics. It may be said to include certain portions of pure mathematics, practical mathematics, theoretical mechanics and, in rare cases, such practical mechanics as is considered necessary to illustrate the theory—whereas, in fact, it should cover all applications of mathematics to the various fields of knowledge.

In technical education to-day the most important application of mathematics is to the field of engineering. Here it is a sign of the times that the University of London has quite recently made mathematics compulsory in the B.Sc. Part I degree in Engineering. Engineering, however, is by no means the only field which comes under the purview of the mathematician in a technical college. Physical chemistry is becoming increasingly mathematical, and, besides the needs of the chemists, we have to consider the applications to physics, optics, architecture, insurance, and handicrafts, while we are interested in, though not always directly concerned with applications to building, commerce, pharmacy, navigation, and art. With regard to those students with whom we are directly concerned, there are roughly three times as many evening students as there are full-time day students. (In many other institutions the proportion is much higher.) A large number of these evening students study "practical mathematics", which may be defined as an elementary course in pure mathematics leading up to and including the calculus and a few special types of differential equations. As much academic mathematics as possible is introduced and constant reference is made to engineering applications. Some teachers define "practical mathematics" as mathematics taught practically, others as the type of mathematics which is most helpful to the allied or more practical subjects. In technical institutions the entry for ordinary pure and applied mathematics is small compared with that for practical mathematics.

Since Perry broke away from the academic tradition, the applications of mathematics have increased so much that the term which originally referred to engineering cannot now be expected to cover

* A paper read at the Annual Meeting of the Mathematical Association, January 1939.

† "Clarity is not enough", by Professor L. Hogben, *Mathematical Gazette*, May 1938.

all applications including those already mentioned and others such as statistics, artillery, economics, power production, genetics, psychology and biomathematics. Perhaps " applied " mathematics would meet the situation if ordinary applied mathematics were renamed as theoretical mechanics and mathematical physics.

II. The Treatment of Mechanics.

If Perry's break from Euclid was definite, how much more so was the break from theoretical mechanics. No system of practical mathematics exists without some form of geometry, and this subject is generally taught by the mathematics department, whereas mechanics is taught, in the early stages, as part of a course called engineering science, which is largely a compound of mechanics, physics, and engineering. The work is done under great difficulties. Not only is the instruction given to students who have already undergone the fatigue of a full day's work, but only one night a week can be devoted to it. The previous education of the evening student is generally less than that of the day student, while his opportunities for homework and private study are often more restricted. Consequently engineering teachers have found it necessary to depart considerably from the ordinary academic method of presenting elementary mechanics. As in the case of congruence theorems in geometry, all difficulties are placed firmly in the background and an attempt is made to reach a working compromise and bridge the gap between academic mechanics and the mechanics of practical experience. " Too long ", says Benchara Branford, " has the mechanics of the academic been divorced from the practical activities of the craftsman." The trouble is, however, that if we are to prove all things all the way, then the bridging of the gap is almost impossible, as, apart from the poor calibre of the students, the mechanics of practical experience is generally much more difficult than so-called academic mechanics. Students also have real difficulties in understanding mechanics and in interpreting the wording of many questions which are set from time to time. Many of them have only had an elementary school education and have little or no knowledge of algebra and geometry. The modern form of mechanics dates back only to the seventeenth century, and even the story of the evolution of simple machines seems to have been lost beyond all hope of recovery.

Despite these difficulties, the attempt to provide a suitable course has met with some success. The approach is generally by ideas of velocity-ratio and " leverage ". Much trigonometry and calculation has to be avoided by drawing and measurement. The ordinary work on the resolution of forces can be omitted and the resultants obtained graphically. In the cases where the forces do not meet in a point, the resultant can be drawn in position by the link polygon or by taking moments. In simple problems on equilibrium the results are obtained from the force diagram. Here the principles of (1) vector

addition, (2) moments, (3) a sort of virtual work (as a basis for work in future years) are emphasised and the method of sections in frameworks is regarded as an example of equilibrium of forces. The theorems which take up much space in ordinary textbooks are omitted as tending to obscure the real issues in the minds of these students, whose outlook is directed towards results and who are not concerned with the difficulties on the way towards them.

In dynamics a good deal of work is done on graphs such as the velocity-time diagram and the construction of the velocity-time and acceleration-time curves from the distance-time and velocity-time curves respectively. The technical student is generally familiar with the type of velocity-time diagram which occurs in actual practice, such as is given on p. 39 of the last report of the Association on the Teaching of Mechanics. More attention is given to correct allowance for scales in these diagrams and a higher standard in neatness of drawing is expected from him. A really good draughtsman has been known to draw 20 separate lines in the space of one-fifth of an inch.

It is needless to add that the treatment of machines is more detailed, and terms like mechanical advantage and velocity-ratio are frequently employed. Engineering students are trained to use the lb.-wt. units and errors in the "superfluous g" are perhaps not quite so common. The work on wire testing, stress and strain, modulus of elasticity, torsion of wires, shear, bending moments, deflection of beams is also more detailed and the methods are generally experimental and graphical. Friction is approached experimentally; cams are studied under mechanisms and the loci are done by drawing; and some hydraulics is also done.

In the dynamics of rotation and rigid bodies practically no attempt is made to prove the fundamental theorems involved. The results are merely quoted and the students rarely dream of questioning them. In any case they have not the equipment necessary to understand the proofs in detail, as even the sign Σ worries them a good deal. Hence they assume straightaway that, for its translational motion, a rigid body moves like a particle as if all its mass were concentrated at its centroid, and for its rotational motion as if the centroid were fixed and the body rotated about it. So they proceed, appealing either to common sense as in the case just mentioned, or to experiment, or to drawing when they wish to avoid troublesome mathematics. Now and then the use of empirical formulae enables them to proceed as far as gyroscopic problems. Some of these problems, which would prove difficult for the mathematical specialist, are accepted quite coolly by the technical student, very much as the student who accepts the formula for the periodic time of a compound pendulum from an experimental verification, without any attempt to justify the law. In a mechanism the moments of inertia can sometimes be simplified by considering the equimomental system. The technician is never worried as to why it is the equimomental system. He is only too thankful to know that he has found a short-cut for his problem. Later in the theory of vibrations

(including Rayleigh's principle), assumptions and empirical formulae are widely used.

In the technical courses the use of the instantaneous centre of rotation occurs earlier than in the ordinary school courses of mechanics. The difference in outlook might be illustrated by the following example. Two solutions are given, with and without the use of the instantaneous centre.

Consider a four-bar mechanism $ABCD$ (Fig. 1). In it AB is fixed and D and C can turn about A and B respectively. $AB=3$ units, $AD=1$, $DC=2$, $CB=2\frac{1}{4}$, $BK=1$. Find the velocity of K, given that the velocity of D is 3 units when $\theta=90°$.

The pupil who had not done instantaneous centres might tackle the problem as follows :

By projection perpendicular to AB

$$\sin\theta+2\sin\phi=2\tfrac{1}{4}\sin\psi. \quad\text{......................(1)}$$

By projection parallel to AB

$$\cos\theta+2\cos\phi+2\tfrac{1}{4}\cos\psi=3. \quad\text{...............(2)}$$

Using $CD^2=CE^2+ED^2$ and putting $\theta=90°$, we get

$$(2\tfrac{1}{4}\sin\psi-1)^2+(3-2\tfrac{1}{4}\cos\psi)^2=4,$$

which reduces to

$$59/24=\sin\psi+3\cos\psi. \quad\text{...........................(3)}$$

Differentiating (1) and (2) and putting $\theta=90°$, $\dot{\theta}=3$, we have

$$2\cos\phi\,.\,\dot{\phi}=2\tfrac{1}{4}\cos\psi\,.\,\dot{\psi},$$

$$3+2\sin\phi\,.\,\dot{\phi}+2\tfrac{1}{4}\sin\psi\,.\,\dot{\psi}=0.$$

Eliminating $\dot{\phi}$, $\dot{\psi}=-4\cos\phi/3\sin(\phi+\psi)$.

Solving (3) we find ψ is approximately 57° 12′ and so ϕ is approximately 25°. Hence $\dot{\psi}=-1{\cdot}2$ approximately and the velocity of K is 1·2 units of velocity.

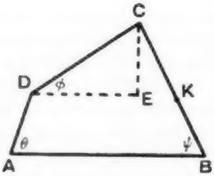


FIG. 1.

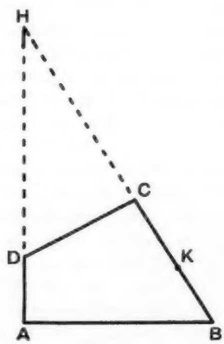


FIG. 2.

The technical student would probably proceed thus. In Fig. 2 produce BC and AD to meet in H, which is the instantaneous centre for CD. He would then *measure* HC and HD; $HC = 3{\cdot}25$, $HD = 3{\cdot}6$.

Hence (velocity of C) : (velocity of D) $= HC : HD$

$$= 3{\cdot}25 : 3{\cdot}6.$$

Thus the velocity of C $\qquad = 3 \times 3{\cdot}25/3{\cdot}6 = 2{\cdot}7$ units,

and the velocity of K $\qquad = 2{\cdot}7 \times 1/2{\cdot}25 = 1{\cdot}2$ units.

Sometimes he solves the problem also graphically by means of a velocity diagram. The theory behind this method is as follows:

Let A, B, C be three points of a body (Fig. 3), and suppose that the velocity of A is known as V_A in magnitude and direction. Let the angular velocity be ω, measure along AP at right angles to V_A a distance PA equal to V_A/ω, then P is the instantaneous centre.

The velocity of B, $V_B = \omega \,.\, PB$; of C, $V_C = \omega \,.\, PC$.

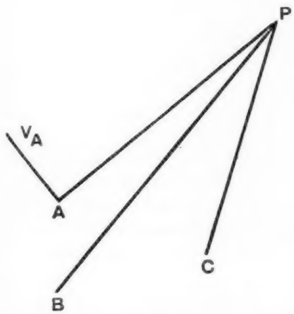


FIG. 3.

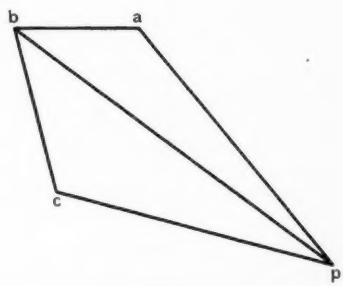


FIG. 4.

These are obtained graphically by drawing pa to represent V_A in magnitude and direction (Fig. 4); ab is drawn at right angles to AB, pb to PB, bc to BC, pc to PC.

$\triangle pab$ is similar to $\triangle PAB$ so that if $V_A = \omega \,.\, PA$, since

$$pb/pa = PB/PA,$$

then $\omega \,.\, PB = V_B$.

Similarly $\triangle pbc$ is similar to $\triangle PBC$ so that pc represents V_C to the same scale as pa represents V_A.

Thus we have a sort of velocity diagram, somewhat similar to the polygon of forces, which enables us to obtain graphically and approximately the velocity of a carried point of a mechanism when ordinary analytical methods are often long and involved.

For example, in the four-bar mechanism question worked above, pa ($= 3$ units) is drawn at right angles to AD (Fig. 5), ab is drawn at right angles to DC, pb represents the velocity of C (pb at 90° to BC).

By measurement $pb = 2{\cdot}7$ units.

The velocity of K as before $= 2{\cdot}7 \times BK/BC = 1{\cdot}2$ units.

In another connection the use of the instantaneous centre enables us to obtain a simple proof for the kinetic energy of a rigid body, which might be noticed in passing, although it is not really relevant here.

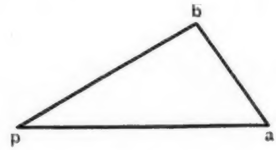


FIG. 5.

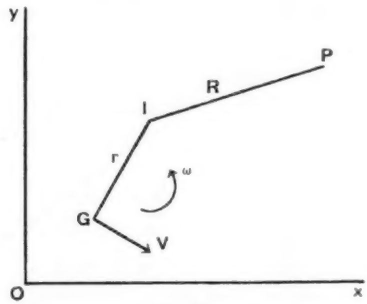


FIG. 6.

Let G be the centroid and V its velocity. Let the angular velocity be ω. Let I be the instantaneous centre and let the element of mass m be taken at P distant R from I (Fig. 6).

$$\begin{aligned}\text{Kinetic energy} &= \Sigma \tfrac{1}{2} m R^2 \omega^2 \\ &= \tfrac{1}{2}\Sigma m\omega^2 (r^2 + PG^2 - 2r \,.\, PG \cos PGI) \\ &= \tfrac{1}{2} M V^2 + \tfrac{1}{2}\omega^2 \Sigma m PG^2 \,;\end{aligned}$$

since $V = r\omega$, $\Sigma m = M$ and $\Sigma m PG \cos PGI = 0$ as the first moment about an axis through G is zero ; that is,

$$\text{kinetic energy} = \tfrac{1}{2} M V^2 + \tfrac{1}{2} M k^2 \omega^2,$$

where k is the usual radius of gyration.

Enough has been said to show, I think, that the problem of presenting mechanics in this way is that in which the minimum knowledge of elementary mathematics can be assumed. Although it has grave disadvantages from the mathematical point of view, it might help when properly arranged to provide a course in schools of the non-specialist type for pupils spending an odd year at school. On the other hand, the technical student might not be driven so

much to the drawing board if more algebra and geometry and less arithmetic were done in the Junior or Elementary School courses. The efficiency of these courses in mechanics evidently awaits improvements in the mathematical groundwork, and similarly with regard to ideas of geometry and practical drawing.

Of course the better students do eventually take courses in mechanics in the mathematical department, and it seems agreed that, in their case, there is room for both theoretical mechanics as well as applied mechanics. This remark applies to less than half of our full-time students. In the evening department the percentage of those taking theoretical mechanics is as low as 5%. In these classes the range of work is from matriculation to final B.Sc. standard in applied mathematics, and they are governed largely by the appropriate examination syllabuses. The teachers generally prefer to teach mechanics without using any apparatus and appeal to intuition rather than experiment. Personally I agree with the late Professor Filon, who said that although "it would be a great mistake to transform a course of applied mathematics into one of experimental by dragging in a laboratory experience which the student does not need, yet enough should be given him to impress him with the significance of the problems which are set before him". So we have the part use of a small laboratory which can be used to illustrate the theory whenever possible. We are definitely hampered, however, by lack of time. Hence we have to concentrate on essentials and use short methods wherever possible. The approach is generally through straight-line motion, then from the parallelogram of accelerations to the parallelogram of forces, through statics and then back to dynamics, finishing with hydrostatics.

III. The Future of "Applied" Mathematics.

I have endeavoured to define "applied" mathematics, and because I was asked to do so, I have recorded a few impressions about mechanics. Judging from criticisms on classes under our control, there seems to be a definite movement towards making the teaching of mathematics conform more to the problems of everyday life. It is hoped that the teacher will spend some of his time in making up problems in arithmetic, for example, from the needs of shopkeepers, craftsmen and people with whom he comes in contact.

The following problem arose in somewhat similar circumstances: What percentage increase in turnover would be required if a traveller cut his prices, by say r_1 per cent., when working on a margin of r_2 per cent.?

Here the percentages are on the selling price. Assuming that he must increase his sales by R per cent., then

$$1+\frac{R}{100}=\frac{r_2}{r_2-r_1}$$

$$i.e. \quad R=\frac{100r_1}{r_2-r_1}. \quad \dots\dots\dots\dots\dots\dots(1)$$

(i) For a given r_2, the R, r_1 curve is a hyperbola. For different values of r_2 we have different hyperbolae from which can be read off the requisite percentage increase in sales to balance a proposed percentage cut in prices.

(ii) Rewriting (1) in the form $r_2/r_1 = 1 + 100/R$, for given separate values of r_2 we might obtain a series of straight lines by plotting $1/r_1$ against $1/R$.

(iii) Substituting $z = 1 + 100/R$, the formula becomes $r_2 = r_1 z$. We can now construct a line chart or nomogram, using logarithmic scales, for $r_2 = r_1 z$. The scales would be labelled in terms of r_2, r_1, R so that a straight edge, joining r_2 on one scale to r_1 on another would intersect the third scale in R. This would probably be the most convenient form for a salesman who was only interested in results. A few examples would suffice to show him the enormous work involved to counteract any price cutting. For example, if he works on a margin of 15 per cent. profit and he cuts his prices by 5 per cent., his turnover must be increased by 50 per cent. (Fig. 7).

Here we not only solve a particular question, but endeavour to show how such a series of questions might be quickly evaluated in practice.

At the other end of the field of " applied " mathematics we have imposed upon us the need for development by the ever increasing boundaries of science. As science progresses either by graphical methods or experimentally or empirically, it should be possible to formulate a theory filling in the gap. The mathematics involved will not, in the first place, be pretty mathematics. I have heard it aptly described by Vaughan Johnston as Brute-Force mathematics. Nowhere was this better illustrated than in the field of technical optics. Sir William Hamilton then invented the Characteristic Function and produced order into the chaos of the general theory of geometrical optics. Apart from one lens system, however, it has been left to the technician to adapt mathematics for his purpose in tracing rays through an optical system, by which, with optical intuition, he can design and improve optical systems in various instruments. To a mathematician the beauty of Hamilton's work is so apparent that he can but feel for the discords and clashings in the work of the technicians. With the spirit of Herschel, who said that " Numerical precision is the soul of science ", these technicians are prepared to plough their way through any number of calculations, however formidable, in order to achieve their purpose and obtain the answer.

In another field of enquiry and as a small example of this attitude, I should like to quote the following example. It should at least show the value of the liaison between the mathematician and technician. It relates to the *ungula*.

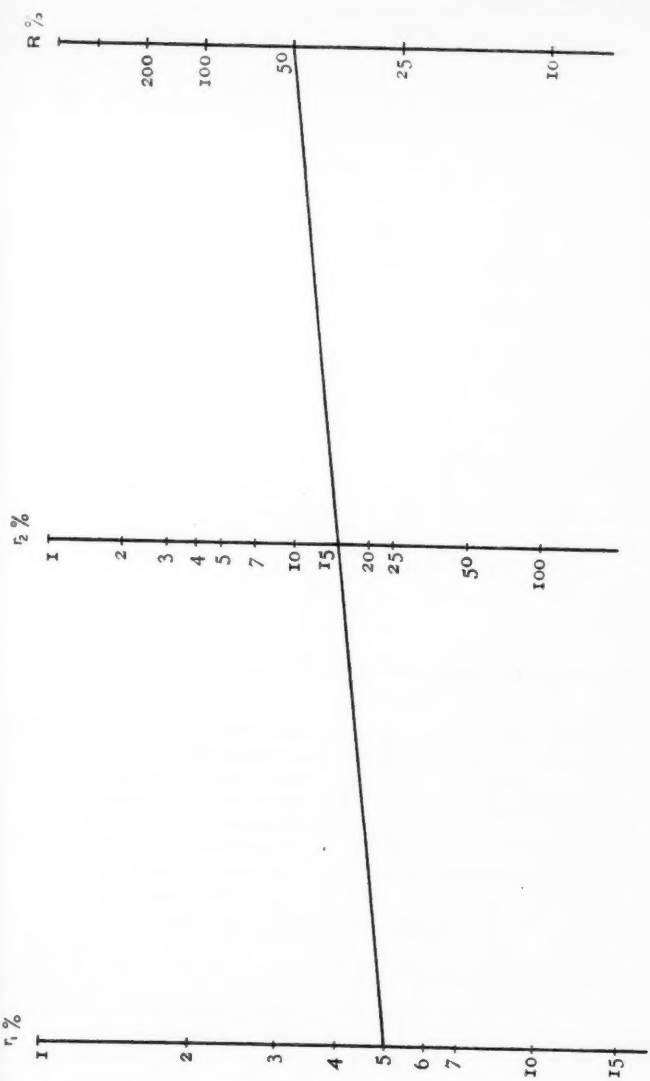


FIG. 7.

THE UNGULA.

The ungula is a portion cut off from pyramids, prismoids, cylinders and cones by a plane section, which is inclined to the base. The problem under our consideration related to such a plane section of a cylinder.

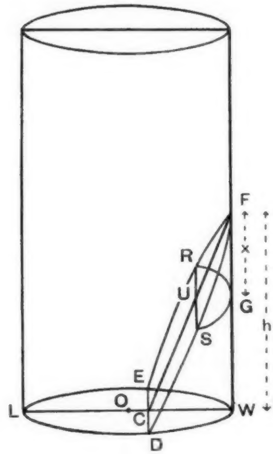


FIG. 8.

Let the plane section have a slope of m, that is,

$$FW = h = m \cdot CW = mk,$$

and cut the base of the cylinder in DE (which is at right angles to the diameter LW) and the generator of the cylinder through W in F. Consider a section of the ungula RSG parallel to its base EDW cutting the generator through W in G.

Let $FG = x$, $UG = H$; then $H/k = x/h$ (Fig. 10) or $x = mH$, $dx = m \cdot dH$.

The volume of the ungula is

$$\int A \cdot dx = m \int A \cdot dH,$$

where A is the area of $RUSG$

$$\begin{aligned} &= \left[mAH - m\int H \cdot dA \right]_{H=0}^{H=k} \\ &= mA_1k - mA_1\bar{z} \\ &= mA_1(k - \bar{z}) \\ &= mA_1\alpha, \qquad \text{...(1)} \end{aligned}$$

where A_1 is the area of DEW, $A_1\alpha$ is its first moment about DE, or α is the distance of the centroid of the segment DEW from DE. ($\int H\,.\,dA = A_1\bar{z}$ gives the first moment about a line through W parallel to DE, $\bar{z}$ is the distance of the centroid from W.) If G_1 is the centroid of the segment DEW (Fig. 9), and $b = \frac{1}{2}DE$,

$$A_1\,.\,OG_1 = \tfrac{2}{3}b^3,$$

or, if r is the radius of the cylinder,

$$A_1(\alpha + r - k) = \tfrac{2}{3}b^3,$$

and the volume of the cylindrical ungula is

$$m\,\{\tfrac{2}{3}b^3 - A_1(r-k)\}. \quad \text{.............................} (2)$$

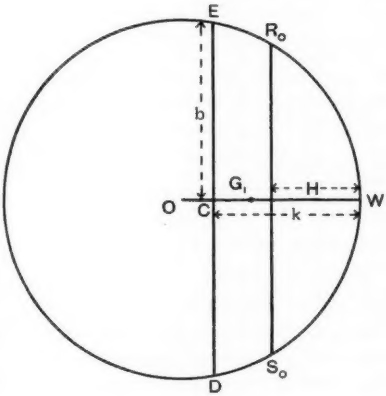


FIG. 9.

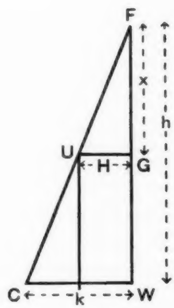


FIG. 10.

In Fig. 9 R_0S_0 is the projection of RS on the base and $R_0S_0 = RS$.

The formula (1), however, applies to a cylinder on any base provided the base has LW as an axis of symmetry.

For a large m, when turned over with its inclined face horizontal, the ungula makes a useful channel. Its maximum volume for a given A_1 will occur when the centroid of the base is as low as possible, and for this purpose the base A_1 is made of circular arcs (instead of a complete circle) with a bulge towards the bottom. The following diagram (Fig. 11) shows the end or base of such a channel. The heights are shown every half-inch from 0 to 13 inches. The lengths of the corresponding chords are given for each height. The areas from the bottom are also given up to each height. The radii of the circles comprising the perimeter are shown. Then the volumes of the ungulas were calculated by formula (2) properly adjusted to allow for the different circles comprising the perimeter and the different heights. These figures are not reproduced here.

These calculations must have been long and tedious. They were evidently initiated without a knowledge of the simple formula (1). A student of mathematics could have explained that the volume of the ungula is the product of the slope (m) and the first moment of the segment of the base or end about DE ($A_1\alpha$), so that for this purpose one need only consider the end or base of the ungula once you have multiplied by m. Hence the calculations could have been obtained much more quickly by the use of the planimeter for areas and the integrator for first moments, or, failing the use of the integrator, the centroids could have been obtained by cutting out in cardboard and using the ordinary method of suspension.

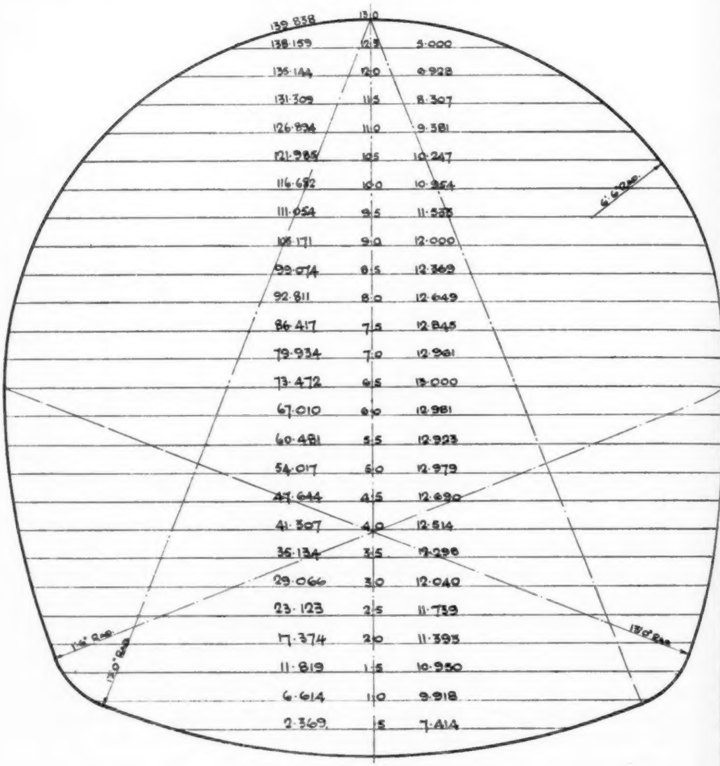


FIG. 11.

Depths shown in centre column.
Chords shown in right column.
Areas shown in left column.

This example illustrates, I think, that spirit of mathematical economy which obviates waste of labour on what can be readily systematised. The future of technical mathematics, therefore, seems to lie in further bridging the gaps between mathematicians and technicians, so that the strength of both may be concentrated on what is known but not systematised and on what is unknown. There will not be competition between mathematicians and technicians, since the latter have generally specialised in one particular branch of science, so much so that they have acquired a sort of second sight or instinct in it which the mathematician cannot hope to acquire as he traverses so many fields in his quest for simplification and order. On the other hand, the technician must sympathise with the mental motives of the mathematician, which often demand such satisfaction in thought that there is a danger of the original purpose of the enquiry being relegated to the background.

One might mention many gaps which are waiting to be filled between the mathematician and the theoretical technician. Progress, however, cannot be made unless a greater army of workers is employed and more time is allowed for research in the Mathematical Departments concerned. Meanwhile, if technicians were less concerned with the teaching of mathematics and more with the progress of their own science and the bridging of the gaps between their theory and practice, the lot of the technical mathematician, who is at present perched between two stools, might be improved.

The relations between mathematicians and technicians have been coloured too much perhaps by the legacy of ancient Greece. Plato was apparently greatly indignant when mechanical instruments were employed for solving geometrical problems, for, says Plutarch, the method absconds from intellectual to sensible things and, besides, employs such bodies as require much vulgar handicraft. Archimedes also held this view in theory but introduced certain inventions in practice. Locke went to the other extreme in declaring that no education was of value if it were not directly useful. Somehow or other the mathematics in technical education must not be any less "education" for having a technical bias. The topics must cease to be a collection of useful results and rough rules, and must become a closely coordinated structure of knowledge.

A. B.

1273. But when the traditional Logic deals with inference, its concern for cogency leads it to treat all inference according to the mathematical model, and in effect to reduce all thinking to mathematics. . . . In its essence it is either arithmetical or geometrical. It is obscurely yet truly an application of the science of Quantity; and the name of that science is Mathematics.—William Temple, *Nature, Man and God.* [Per Mr. A. F. Mackenzie.]

1274. To Plato its [Mathematics'] great value was that it effected a transition away from the physical world, which by its help we might learn to leave behind; it was the science of extended form as apprehended by the pure intellect. But for the Cartesian Mathematics is a study of the extended world.—William Temple, *Nature, Man and God.* [Per Mr. A. F. Mackenzie.]

RATIONAL TRIANGLES.

BY R. L. GOODSTEIN.

1. Triangles with integral sides and circumradius.

A general formula giving such triangles was stated by Gauss; we propose to find a general formula giving prime solutions—that is, a formula giving integral sides and circumradius, a, b, c, R, which have highest common factor unity.

If α, β, γ are the angles of an integral triangle, then $\sin\alpha$, $\sin\beta$, $\sin\delta$ and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are rational.

Let $\sin\alpha = p/q$; then $\sqrt{(q^2-p^2)}$ is rational, so that $q = m^2+n^2$ and $p = 2mn$.

Thus $\quad \sin\alpha = 2mn/(m^2+n^2), \quad \sin\beta = 2pq/(p^2+q^2),$

and
$$\sin\gamma = \sin(\alpha+\beta) = \frac{2(mq+np)(mp-nq)}{(m^2+n^2)(p^2+q^2)}$$

and so
$$a = 4mnR/(m^2+n^2),$$
$$b = 4pqR/(p^2+q^2),$$
$$c = 4(mq+np)(mp-nq)R/(m^2+n^2)(p^2+q^2).$$

If m and n are relatively prime, m^2+n^2 and mn are relatively prime, since $m^2+n^2 = (m+n)^2-2mn$. Similarly p^2+q^2 and pq are relatively prime.

Writing $mq+np = s$, $mp-nq = t$, we see that
$$(m^2+n^2)(p^2+q^2) = s^2+t^2;$$
and so, when s and t have no common factor,
$$(mq+np)(mp-nq) \text{ and } (m^2+n^2)(p^2+q^2)$$
are relatively prime.

Thus when s and t are relatively prime, since c is integral,
$$R = (m^2+n^2)(p^2+q^2)/\epsilon,$$
where $\epsilon = 1$ if m and n are not both odd and p and q are not both odd;

$\epsilon = 2$ if m and n are both odd but p and q are not both odd, or *vice versa*.

m, n, p, q are not all odd since this would make $mq+np$ and $mp-nq$ both even.

If r and s have highest common factor $\lambda > 1$, so that $r = \lambda\rho$ and $s = \lambda\sigma$, where ρ and σ are relatively prime, then
$$\frac{mq+np}{mp-nq} = \frac{\rho}{\sigma},$$
i.e.
$$m(p\rho - q\sigma) = n(p\sigma + q\rho). \quad \text{.........................(1)}$$

Since m and n are relatively prime, it follows that $(p\sigma+q\rho)/m$ and $(p\rho+q\sigma)/n$ are integers; but (1) may be written in the form
$$p(m\rho - n\sigma) = q(n\rho + m\sigma)$$

and p, q are relatively prime, so that $(m\sigma + n\rho)/p$ and $(m\rho - n\sigma)/q$ are integers.

From the equations $$mp - nq = \lambda\sigma, \qquad (2)$$
$$mq + np = \lambda\rho \qquad (3)$$

we have $$\frac{m}{q\rho + p\sigma} = \frac{n}{p\rho - q\sigma} = \frac{\lambda}{2mn}$$

and $$\frac{p}{m\sigma + n\rho} = \frac{q}{m\rho - n\sigma} = \frac{\lambda}{2pq}.$$

Hence $2\,mn$ and $2pq$ are both divisible by λ.

Further, multiplying (2) by m and (3) by n and adding :

$$p(m^2 + n^2) = \lambda[m\sigma + n\rho],$$

and multiplying (2) by n and (3) by m and subtracting :

$$q(m^2 + n^2) = \lambda[m\rho - n\sigma].$$

Hence since λ cannot divide both p and q (for p, q are relatively prime), λ divides $m^2 + n^2$. And similarly λ divides $p^2 + q^2$.

Hence λ^2 divides each of the four expressions $4mn(p^2 + q^2)/\epsilon$, $4pq(m^2 + n^2)/\epsilon$, $4(mq + np)(mp - nq)/\epsilon$, $(m^2 + n^2)(p^2 + q^2)/\epsilon$, so that the general formula for prime triangles is :

$$a = 4mn(p^2 + q^2)/\epsilon\lambda^2,\ b = 4pq(m^2 + n^2)/\epsilon\lambda^2,\ c = 4(mq + np)(mp - nq)/\epsilon\lambda^2,$$
$$R = (m^2 + n^2)(p^2 + q^2)/\epsilon\lambda^2,$$

where λ = highest common factor of $(mq + np)$ and $(mp - nq)$;

$\epsilon = 2$ if m and n are both odd but p and q are not both odd, or *vice versa* ;

$\epsilon = 1$ otherwise.

2. To find three or more rational right-angled triangles with equal areas.

Solutions have been given by Diophantus, Fermat, Euler and others.

The problem is equivalent to the following :

2.01. To find three values of z such that $z^2 \pm k$ are both squares ; for if $x^2 + y^2 = z^2$ and $xy = k/2$, then $z^2 \pm k$ are both squares.

2.02. To find three squares in arithmetical progression, with common difference k ; for if $x^2 - y^2 = k$, $y^2 - z^2 = k$, then $y^2 \pm k$ are both squares. The equations $x^2 - y^2 = y^2 - z^2 = k$ lead to $x^2 + z^2 = 2y^2$, which is solved by taking $p = \frac{1}{2}(x + z)$, $q = \frac{1}{2}(x - z)$, giving $p = u^2 - v^2$, $q = 2uv$, $y = u^2 + v^2$ and hence $x = u^2 - v^2 + 2uv$, and the condition $x^2 - y^2 = k$ leads back to $pq = k/2$.

2.03. With the equation $x^2 + z^2 = 2y^2$ is related the problem of automedian triangles, whose medians are proportional to their sides.

We derive two formulae, one giving an infinity of trios of rational right-angled triangles with equal areas and the other giving an infinity of such triangles all having the same area.

2.1. We first find integral solutions of the equation

$$pq(q^2-p^2)=rq(q^2-r^2),$$

i.e. $$q^2(p-r)=p^3-r^3,$$

i.e. $$q^2=p^2+pr+r^2=(rn-p)^2 \text{ say},$$

i.e. $$p(2n+1)=r(n^2-1).$$

2.11. Take $p=n^2-1$, $r=2n+1$, then $q=n^2+n+1$.

Furthermore, $rq(q^2-r^2)=rq(p^2+pr)=q(p+r)\{(p+r)^2-q^2\}$.

Hence the three triangles of sides

$$q^2-p^2, \quad 2pq, \quad q^2+p^2;$$
$$q^2-r^2, \quad 2qr, \quad q^2+r^2;$$
$$(p+r)^2-q^2, \quad 2q(p+r), \quad (p+r)^2+q^2$$

have the same area, where p, q, r have the values given in 2.11.

If we put $n=h/k$ in 2.11 and multiply the sides of the triangles by k^4, we obtain a doubly infinite set of trios.

2.2. Let u, v, x be the sides of a right-angled triangle of area $a/4$. Then $u^2+v^2=x^2$ and $uv=a/2$.

Hence x^2+a and x^2-a are both squares.

2.201. Furthermore, if $x^4-a^2=y^2$, where x and y are relatively prime, then x^2+a, x^2-a have no common factor (for a common factor would be a factor of (x^2+a+x^2-a), *i.e.* of x and also of y), and so $x^4-a^2=y^2$ implies that $x^2\pm a$ are both squares.

2.202. Let $x^2+a=(x+m)^2$ and $x^2-a=(x-n)^2$, so that

$$x=(a-m^2)/2m \text{ and } x=(a+n^2)/2n,$$

whence $$a(n-m)=mn(m+n);$$

put $a=\lambda mn$, $m+n=\lambda(m-n)$; and so $m=n(\lambda-1)/(\lambda+1)$.

Therefore $$a=\frac{\lambda(\lambda-1)}{\lambda+1}n^2=\lambda(\lambda^2-1)\left(\frac{n}{\lambda+1}\right)^2.$$

Put $k=(\lambda+1)/n$, so that $\lambda(\lambda^2-1)=ak^2$.

Put $\lambda=a\mu^2$ and $k=\mu k'$, and then

$$a^3\mu^4-1=k'^2.$$

Put $\xi=a\mu$, $\eta=ak'$ and we have

2.21. $$\xi^4-a^2=\eta^2.$$

Now $\lambda=\xi^2/a$ and $k=\xi\eta/a^2$, so that $n=a(\xi^2+a)/\xi\eta$ and hence

$$\begin{aligned} x &= \{\xi^2\eta^2+a(a+\xi^2)^2\}/2\xi\eta(\xi^2+a) \\ &= \{\xi^2(\xi^4-a^2)+a(\xi^4+2a\xi^2+a^2)\}/2\xi\eta(\xi^2+a) \\ &= \{\xi^4+a^2\}/2\xi\eta \end{aligned}$$

and $$y=\{\eta^4-4a^2\xi^4\}/4\xi^2\eta^2.$$

Thus from one solution of 2.21 we obtain another, and from this a third, and so on as often as we please.

One solution of 2.21 is:

$$\left.\begin{aligned} a&=4\alpha\beta(\alpha^2-\beta^2),\\ \xi&=\alpha^2+\beta^2,\\ \eta&=(\alpha^2-\beta^2)^2-4\alpha^2\beta^2.\end{aligned}\right\}\ \alpha,\ \beta \text{ integers, } \alpha>\beta.$$

To show that all the solutions of 2.21 obtained by this method are different, we shall examine under what conditions $x>\xi$.

We require $$\{\xi^4+a^2\}/2\xi\eta>\xi,$$

i.e. $$\xi^2(\xi^2-2\eta)+a^2>0,$$

which is satisfied if $\xi^2\geqslant 2\eta$—that is, if $(\alpha^2+\beta^2)^2\geqslant 2\{(\alpha^2-\beta^2)^2-4\alpha^2\beta^2\}$—that is, if $14\alpha^2\beta^2\geqslant\alpha^2+\beta^2$, which is obviously satisfied if α and β are $\geqslant 1$.

There remains to show that $\xi^2\geqslant 2\eta$ implies $x^2\geqslant 2y$; we require

$$(\xi^4+a^2)^2\geqslant 2\eta^4-8a^2\xi^4,$$

i.e. $$(\eta^2+2a^2)^2\geqslant 2\eta^4-8a^2\eta^2-8a^4,$$

i.e. $$12(a^2+\eta^2)^2\geqslant 13\eta^4,$$

i.e. $$12\xi^8\geqslant 13\eta^4,$$

which is satisfied when $\xi^2\geqslant 2\eta$.

Thus we have determined an infinity of strictly increasing solutions of equation 2.21. To each solution (x, y) corresponds a rational right-angled triangle of sides

$$\tfrac{1}{2}\{\sqrt{(x^2+a)}+\sqrt{(x^2-a)}\},\quad \tfrac{1}{2}\{\sqrt{(x^2+a)}-\sqrt{(x^2-a)}\},\quad x$$

and of area $a/4$, where $a=4\alpha\beta(\alpha^2-\beta^2)$. R. L. G.

1275. . . . It would follow that the Greeks studied geometrical figures as species of an Idea, conceiving them as related to one another not physically at all, but only intelligibly or logically. But if every geometrical figure is a part of one single space, it follows that geometrical knowledge is knowledge of that single spatial system which comprises (or even which *is*) the physical universe.—William Temple, *Nature, Man and God*. [Per Mr. A. F. Mackenzie.]

1276. MATHEMATICS AND RELIGION.

It may reassure those who, like the Archbishop of Canterbury, are disconcerted at the intrusion of mathematics into the higher regions of speculation to read the preface once written (or suggested) by the late (*sic*) G. H. Hardy to a book on pure mathematics: "This subject has no practical value—that is to say, it cannot be used to accentuate the present inequalities in the distribution of wealth, or to promote directly the destruction of human life."—T. C. P., Leeds, Letter in *The Observer*, December 25, 1938. [Per Mr. A. F. Mackenzie.]

1277. . . . The editor . . . defends nonsense as a product of the imagination. "Nonsense", he writes, "is the name given by knowledge to imagination's account of its travels in regions uncharted by practical experience". This is a pretty big claim. It might also be a definition, for example, of mathematics. —From a review by A. G. MacDonnell in *The Observer*, December 25, 1938. [Per Mr. A. F. Mackenzie.]

CHINESE UNIT OF LENGTH IN THE EARLY CH'ING DYNASTY.

KWAN CHAO CHIH.

INTRODUCTION.

A little to the north of the Yenching University near Peking stand the ruins of the famous Yuan Ming Yuan. It was a palace built during the reign of Emperor K'ang Hsi of the Ch'ing Dynasty, about 200 years ago, and was given to and repaired and enlarged by Yung Cheng, his son. The Chinese unit of length at the time of building may have been different from the modern unit, and it is therefore interesting, mathematically and archaeologically, to determine the old unit by a mathematical method. Additional interest arises from the fact that the cut stones remaining belong to a section of the palace designed in Western style by Jesuit Fathers who came as missionaries to China. Did they use Chinese or French units in their plans?

METHOD.

The method used in this investigation was similar to that described by G. F. Cramer*.

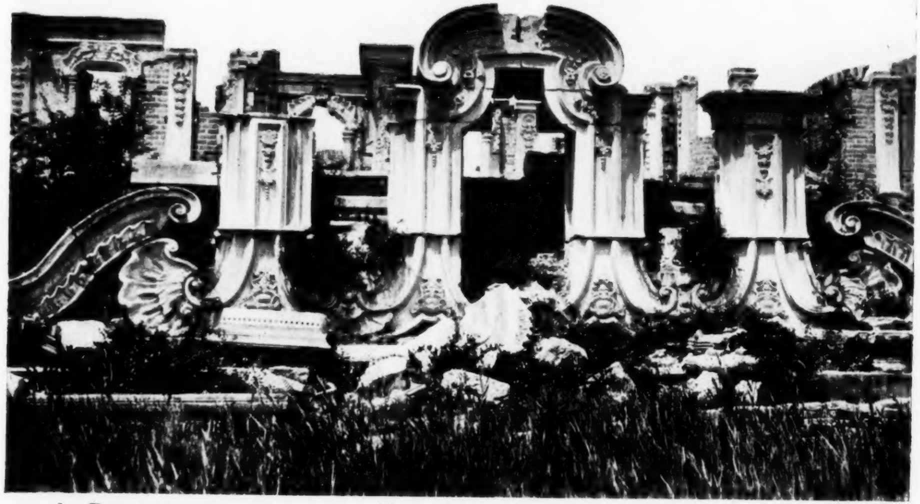
If we take a set of measurements which are not multiples of any unit and plot them along an axis, the points obtained will not cluster: we shall find a random distribution. If we choose any number N and plot its multiples on the same axis, and then lay off on each side of each of this second set of points a small interval of length $n\,(2n<N)$, the points of the first set will as likely fall inside as outside the double intervals $2n$. Therefore approximately $2n/N$ of the random set of points will fall within all the intervals $2n$. We thus define the *Predicted Fraction*, $p=2n/N$.

On the other hand, if the set of measurements consists of approximate multiples of some unit, the points will cluster. In this case, if we can choose N near to the unit or one of its multiples, we shall get a much larger proportion of points inside the intervals $2n$. If M is the total number of points and m the number inside the intervals $2n$, we now define the *Actual Fraction*, $P=m/M$, and the *Ratio*, $R=P/p$. When N is nearly a multiple of the unit we shall have $P>p$ and $R>1$. On the contrary, if N is not nearly commensurable with the unit, $P<p$ and $R<1$. Remarkably large values of R will show that N is very near the unit. Again, if we take $2N$, $3N$, etc., for N, we shall expect large values for R if N is near the unit.

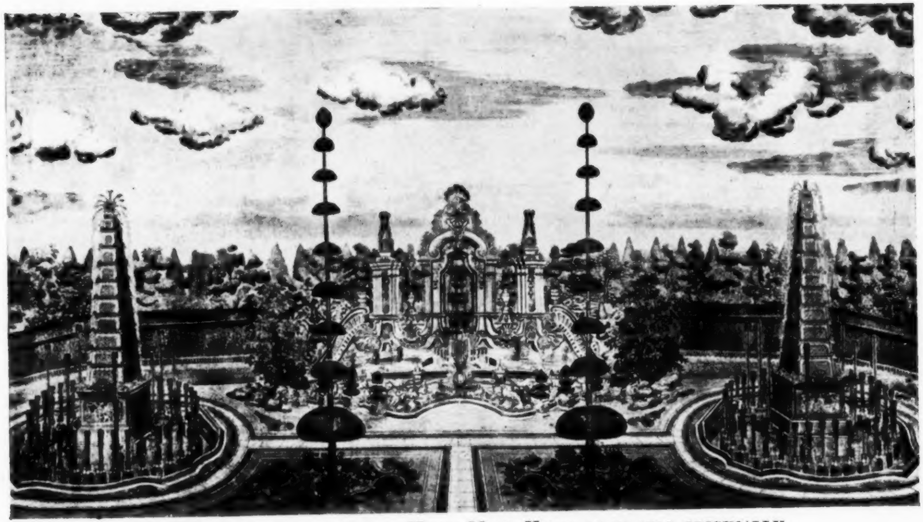
PROCEDURE.

The following plan was recently carried out by students of the Mathematics Department of Yenching University. Eight members were divided into four groups and each group measured dimensions and ornamentation of fallen blocks of stone in a certain region.

* "Determination of a Mayan unit of Linear Measurement", *Amer. Math. Monthly*, June 1938.



1. Ruins of the section of the Yuan Ming Yuan where measurements were made.
(*Plates: Hariangs, Pekin.*)



2. The same part of the Yuan Ming Yuan as it was originally.

In all about 300 measurements were taken, but as it seems likely that the smallest lengths were not measured by the original workmen, and that the largest lengths had large errors, all measurements less than 5 cm. or greater than 150 cm. were rejected. The number of measurements was thus cut down to 233. Checking showed that our errors in measurement were certainly less than 0·5 cm.

RESULTS.

We used the value $n = 0{\cdot}6$ cm.

From the graph the average distance between consecutive pairs of clusters was seen to be nearly 3 cm. Values of N near 3 cm. were therefore tried first. By the use of tracing paper the labour of defining intervals and counting points was lightened. The results were as follows :

N (cm.)	m	R
3·0	102	1·09
3·1	96	1·07
3·2	130	1·49
3·3	103	1·22
3·4	73	0·89
1·6	187	1·07
2·4	113	0·97
6·4	90	2·06
9·6	46	1·58

DISCUSSION.

It can be seen from the table that $N = 3{\cdot}2$, 6·4 and 9·6 all give extraordinarily large values for R. When $N = 1{\cdot}6$, $R = 1{\cdot}07$, so that 1·6 is not likely to be the unit. Values of N near 3·2—namely 3·0, 3·1, 3·3—also give larger values for R, which we should expect if we allow for original inaccuracies on the part of the workmen and effects of weathering. Other values of R taken at random, such as 2·4 and 3·4, give $R < 1$. 3·2 is therefore suspected as the old unit. We prefer it to 6·4 or 9·6 for the following reason.

The standard Chinese foot now used is exactly one-third of a metre, and therefore the Chinese inch (one-tenth of a foot) is about 3·3 cm. But this is recent. In 1907, four years before the Republic, the Manchu Government adopted as standard a foot of 32 cm.* This was taken from the dimensions of the base of an iron vessel made in the forty-third year of the Emperor K'ang Hsi (1704), and was consistent with the figures recorded in contemporary documents.

This unit agrees exactly with the result we obtained. The dense cluster of points near 64 cm. shows that the error of this determination is probably less than 0·1 cm.

We may therefore conclude that the designs, though French, were executed in Chinese units which changed by a negligible amount until the recent Government alteration.

* *History of Chinese System of Weights and Measures,* by Wu Cheng Lo.

APPLICATION OF THE MELLIN INVERSION THEOREM TO IMPULSES.

By N. W. McLachlan.

1. *Introduction.* Impulses occur not only in technical work but in everyday existence. There is hammering and door banging which may cause annoyance, handclapping which denotes enthusiasm and enjoyment, motoring on rough roads which causes discomfort (especially after a good meal!), speech, electrical communication in the Morse code, and so on *ad infinitum.* Impulses have been treated by applied mathematicians with reference to various technical problems, *e.g.* Heaviside in his work on electromagnetic theory. Moreover, the subject is one which merits attention, so in what follows a brief outline will be given of recent work based upon a particular case of the Mellin inversion theorem. This can be stated as follows:

If
$$\phi(p)=p\int_0^\infty e^{-pt}f(t)\,dt, \qquad\text{.......(1)}$$

then will
$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{zt}\phi(z)\frac{dz}{z}, \qquad\text{.......(2)}$$

and vice versa, provided certain conditions are fulfilled.*

The chief conditions required herein are (i) that all the singularities of $\phi(z)/z$ lie on the left of the contour $c\pm i\infty$, and (ii) that

$$|\phi(z)/z|\to 0 \text{ uniformly with respect to phase } z,$$

as
$$|z|\to\infty,\quad -\tfrac{1}{2}\pi\leqslant\text{phase } z\leqslant\tfrac{1}{2}\pi. \qquad\text{.......(3)}$$

This latter condition means that integral (2) taken along an infinite semicircle on the right of $c\pm i\infty$ is zero ($c>0$). When all the essential conditions are fulfilled, (2) is the solution of (1) as an integral equation for $f(t)$, whilst (1) is the solution of (2) as an integral equation for $\phi(z)$.

Two salient cases of impulses are treated herein; (i) where the duration of the impulse must be regarded as >0, (ii) where the duration is so short that it may be considered to $\to 0$, since the result obtained in applications is then sufficiently accurate. In this latter instance the whole of the energy is transferred to the impulsed system so quickly that from a practical viewpoint the transfer of energy may be regarded as taking place instantaneously.

* See McLachlan, *Complex Variable and Operational Calculus* (Cambridge, 1939), where a complete proof of the Mellin theorem and the conditions for validity are given in Appendix 4. In the *Math. Gaz.* 22, 264 (1938), Professor Carslaw has commented upon the presence of p outside an integral of type (1). The reason for the external p is twofold: (a) the operational forms (Laplace transforms) then obtained agree with those used by Heaviside, which are of long standing, (b) if t and p are considered to have dimensions d and d^{-1} respectively, so that the index pt is dimensionless, $f(t)$ and $\phi(p)$ in (1) have the same dimensions, which is useful for checking purposes. See *Phil. Mag.* 26, 394 (1938).

2. *Operational form of a function for a finite interval.*

Instead of the interval $t=0$ to ∞ in (1) we shall consider the interval $t=h_1$ to h_2, $h_1 < h_2$, both quantities being real and positive. We define the O.F. of $f(t)$ for this interval to be

$$\phi(p)_{h_1/h_2} = p\int_{h_1}^{h_2} e^{-pt} f(t)\,dt, \quad \text{......................(4)}$$

thereby implying that we are dealing with a function which is zero outside the range $t=h_1$ to h_2. Repeated integration by parts yields

$$\phi(p)_{h_1/h_2} = \sum_{r=0}^{n-1} p^{-r}[e^{-ph_1} f^{(r)}(h_1) - e^{-ph_2} f^{(r)}(h_2)] + p^{-n+1}\int_{h_1}^{h_2} e^{-pt} f^{(n)}(t)\,dt, \text{......(5)}$$

where $f^{(0)}(u)$ signifies $f(u)$. If, as is usual, we define the O.F. of $f(t)$ for the range $t=0$ to ∞ to be

$$\phi(p) = p\int_0^\infty e^{-pt} f(t)\,dt, \quad \text{........................(6)}$$

then since (by definition),

$$\phi(p)_{0/h} = \phi(p) - \phi(p)_{h/\infty}, \quad \text{........................(7)}$$

by writing $h_1 = h$ and $h_2 = \infty$ in (5), we get

$$\phi(p)_{0/h} = p\int_0^\infty e^{-pt} f(t)\,dt - e^{-ph}\sum_{r=0}^{n-1} p^{-r} f^{(r)}(h) - p^{-n+1}\int_h^\infty e^{-pt} f^{(n)}(t)\,dt, \text{......(8)}$$

provided $e^{-ph_2} f^{(r)}(h_2) = 0$ when $h_2 = \infty$. Putting $h_1 = 0$, $h_2 = h$ in (5) we get a second formula, namely

$$\phi(p)_{0/h} = \sum_{r=0}^{n-1} p^{-r}[f^{(r)}(0) - e^{-ph} f^{(r)}(h)] + p^{-n+1}\int_0^h e^{-pt} f^{(n)}(t)\,dt. \text{......(9)}$$

A third formula can be obtained by adding (8), (9) and halving the sum.

By substituting for $f(t)$ from (2) into (4)—the necessary conditions mentioned in § 1* being satisfied—it can be shown that

$$\phi(p)_{h_1/h_2} = \frac{pe^{-ph_2}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zh_2}\frac{\phi(z)\,dz}{z(z-p)} - pe^{-ph_1}\int_{c-i\infty}^{c+i\infty} e^{zh_1}\frac{\phi(z)\,dz}{z(z-p)}, \text{......(10)}$$

* *Loc. cit.*

where $R(p) > R(z) = c$ on the contour, and the evaluation at the pole $z = p$ is excluded. From (7) and (10) we find that

$$\phi(p)_{0/h} = \phi(p) + \frac{pe^{-ph}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zh}\frac{\phi(z)\,dz}{z(z-p)}, \quad \text{............(11)}$$

the pole at $z = p$ being excluded from the evaluation as before.

3. *Application of formulae in* § 2.

Suppose it is desired to find the response of a dynamical system, having one degree of freedom, to an impulse whose O.F. is given by (8) or (9). Let the differential equation for the system be

$$a_0\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_2 y = \xi(t), \quad \text{....................(12)}$$

where $\xi(t)$ represents the applied disturbance in terms of t the time. Following Bateman, Carslaw, Doetsch, and van der Pol,* we multiply throughout by pe^{-pt}, integrate the l.h.s. from $t = 0$ to ∞, and the r.h.s. from $t = 0$ to h, since $\xi(t) = 0$, $h < t < \infty$.† If the initial conditions are $y = y_0$, $y' = y_1$, when $t = 0$, we obtain ultimately

$$p\int_0^\infty e^{-pt}y\,dt = \frac{\phi_1(p)}{\phi_2(p)} + \frac{\phi_3(p)}{\phi_2(p)} = \psi_1(p) + \psi_3(p), \quad \text{.........(13)}$$

where $\phi_1(p) = \phi(p)_{0/h} \subset \xi(t)$,‡ $\phi_2(p) = a_0p^2 + a_1p + a_2$, and

$$\phi_3(p) = y_0(a_0p^2 + a_1p) + y_1a_0p.$$

If $\psi_1(p)$ and $\psi_3(p)$ are such that the condition at (3) is satisfied, the Mellin inversion theorem can be applied to (13). Thus its solution as an integral equation for y is by (2),

$$y = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}\left[\frac{\phi_1(z)}{\phi_2(z)} + \frac{\phi_3(z)}{\phi_2(z)}\right]\frac{dz}{z}. \quad \text{...............(14)}$$

In many cases $\psi_1(p)$ and $\psi_3(p)$ in (13) may be resolved into partial fractions. When one or more of these, say $\chi(p)$, does not satisfy the condition at (3), the corresponding function of t may be obtained by solving

$$p\int_0^\infty e^{-pt}g(t)\,dt = \chi(p), \quad \text{........................(15)}$$

as an integral equation for $g(t)$, assuming of course that the solution exists.

The first member of integral (14) represents (when evaluated) that part of the solution due to the disturbance, whilst the second member (when evaluated) gives the effect of the initial conditions y_0, y_1. For example, in a mechanical system y_0, y_1 will be the initial displacement and velocity, respectively. In an electrical system y_0 corresponds to the initial charge or quantity of electricity Q_0, and y_1 to the initial current (dQ/dt). $\phi_3(p)$ may be regarded as the initial conditions function.

* See Carslaw, *Math. Gaz.* loc. cit. for references.

† By virtue of this condition the integration is in effect from $t = 0$ to ∞.

‡ The sign $\supset$ means that $\phi(p)_{0/h}$ is the operational form of $\xi(t)$, in this case for the range $t = 0$ to h. It was introduced in *Phil. Mag.* 26, 394 (1938).

4. *An array for the initial conditions function.*

The generalised form of $\phi_3(p)$ for the differential equation

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n y = \xi(t), \quad \dots\dots\dots\dots\dots(16)$$

where $\xi(t)$ may be impulsive or otherwise, can be written down by aid of the following array :

$$\begin{array}{r|ccccc|} & p^n & p^{n-1} & p^{n-2} & & p \\ \hline y_0 & a_0 & a_1 & a_2 & \dots & a_{n-1} \\ y_1 & 0 & a_0 & a_1 & \dots & a_{n-2} \\ \phi_3(p) = y_2 & 0 & 0 & a_0 & \dots & a_{n-3} \\ & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots \\ y_{n-1} & 0 & 0 & 0 & \dots & a_0 \end{array} . \quad \dots\dots\dots(17)$$

Multiply each a by the corresponding y on its left and then by the p immediately above. $\phi_3(p)$ is the sum of all terms so obtained. A similar procedure can be devised for a system of differential equations with constant coefficients. In solving an equation like (16), the briefest procedure is as follows : (i) obtain $\phi_1(p)$ the O.F. of $\xi(t)$ *; (ii) write p for d/dt in the l.h.s.† thereby obtaining $\phi_2(p)$; (iii) form $\phi_3(p)$ the initial conditions function from the array (17). Then the solution takes the form given at (13) – (15). The condition at (3) must, of course, be satisfied for (14) to hold. The integral therein which involves $\phi_3(p)$ expresses, term by term, the contributions due to $y_0, y_1, y_2, \dots y_{n-1}$. Thus after each evaluation, the physical effect of the corresponding initial condition becomes evident.

5. *Impulsive force of finite zero-th moment, but infinite amplitude.*

Passing on to impulses of type (ii), we define that represented by

$$\begin{aligned} y &= f(t); \quad 0 \leqslant t \leqslant h, \quad t \text{ real}, \\ &= 0; \quad h < t < \infty, \quad -\infty < t < 0, \end{aligned}$$

to be

$$\int_0^h f(t)\,dt = A, \text{ a constant}, \quad \dots\dots\dots\dots\dots\dots\dots(18)$$

which may conveniently be regarded as the zero-th moment of the impulsive force. $f(t)$ is a function single-valued and continuous between its discontinuities (if any) where it is finite ; whilst the discontinuities, maxima and minima, are limited in number. The O.F. of $f(t)$ is

$$\phi(p) = p\int_0^h e^{-pt} f(t)\,dt, \quad \dots\dots\dots\dots\dots\dots\dots\dots(19)$$

so, when h is small enough, we may write

$$\phi(p) \simeq p\int_0^h f(t)\,dt = Ap, \quad \dots\dots\dots\dots\dots\dots\dots(20)$$

* From a list of O.F.S. if possible.

† It is not implied that $p \equiv d/dt$. This substitution gives $\phi_2(p)$ immediately and saves labour.

by (18). Hence when $h \to 0$,

$$\phi(p) \to Ap, \quad \text{.............................(21)}$$

and in the limit when $h = 0$,

$$f(t) \supset \phi(p) = Ap. \quad \text{.................................(22)}$$

By virtue of (18) the amplitude of the impulsive force is now infinite.

In certain technical problems although $h > 0$, it is small enough for the O.F. to be regarded as Ap. This O.F. is incompatible with the Mellin theorem because $|\phi(p)/p| = A$ a constant, and the condition at (3) is violated.

6. *Iterated impulses of type* (i).

When this type of impulse is iterated indefinitely ($t > 0$), the corresponding function of t can be expressed as a Fourier series, provided the necessary conditions are satisfied. Let the O.F. of the zero-th period ($t = 0$ to h) of the iterated function be $\phi_0(p)$. Then using the shift factor e^{-ph}, the O.F. of the first period ($t = h$ to $2h$) is $\phi_0(p)e^{-ph}$, and of the nth period $\phi_0(p)e^{-nph}$. Hence the O.F. of the function over the range $t = 0$ to ∞ is

$$\phi(p) = \phi_0(p) \sum_{n=0}^{\infty} e^{-nph}, \quad \text{.........................(23)}$$

provided the series converges. Now $R(p) > 0$ for convergence of (1), and with $h > 0$, $n > 0$, it follows that $|e^{-nph}| < 1$, so

$$\phi(p) = \phi_0(p)/(1 - e^{-ph}). \quad \text{.........................(24)}$$

By the Mellin theorem, subject to the condition at (3) being satisfied, the Fourier series of the iterated function is obtained by evaluating the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}\,\frac{\phi_0(z)\,dz}{z(1 - e^{-zh})}. \quad \text{.........................(25)}$$

If alternate impulses are reversed, the O.F. of the zero-th period ($t = 0$ to $2h$) is

$$\phi_0(p)(1 - e^{-ph}), \quad \text{...........................(26)}$$

$-\phi_0(p)e^{-ph}$ representing the O.F. of the reversed impulse ($t = h$ to $2h$).

Consequently the O.F. of this function iterated indefinitely is

$$\phi(p) = \phi_0(p)(1 - e^{-ph})/(1 - e^{-2ph}) \quad \text{..................(27)}$$

$$= \phi_0(p)/(1 + e^{-ph}) \quad \text{..............................(28)}$$

If the impulse of duration h is followed by a quiescent interval h_1, the O.F. of the zero-th period is still $\phi_0(p)$, but that of the iterated function is

$$\phi(p) = \phi_0(p)/[1 - e^{-p(h+h_1)}]. \quad \text{....................(29)}$$

When alternate impulses are reversed,

$$\phi(p) = \phi_0(p)/[1 + e^{-p(h+h_1)}]. \quad \text{....................(30)}$$

The Fourier series corresponding to (28)–(30) are found as shown at (24), (25), provided the Mellin theorem is applicable.

7. *Iterated impulses of type* (ii).

Here $$\phi_0(p) = Ap, \qquad (31)$$
and $$\phi(p) = Ap/(1 - e^{-ph}), \qquad (32)$$

h being the period. Neither (31) nor (32) complies with the condition at (3), so the Mellin theorem is inapplicable. In practical applications, however, if one or more of the inherent forces of the system depends upon a time derivative,* the use of the Mellin theorem is permissible. By way of a simple illustration, take the hypothetical case of a heavy mass m resting on a frictionless horizontal plane. Commencing at $t=0$ a rhythmic series of impulsive blows is given to the mass (say with a hammer), the interval between blows being h. Find the velocity of the mass at any time $t > 0$.

The differential equation for this case is

$$m\frac{dv}{dt} = f_1(t), \qquad (33)$$

where v is the velocity of the mass, and $f_1(t)$ the functional form of the impulsive forces. Proceeding as in §(4), we write p for d/dt and the O.F. (32) for $f_1(t)$. Then we get

$$v \supset A/m(1 - e^{-ph}), \qquad (34)$$

as the operational form of solution, or alternatively,

$$p\int_0^\infty e^{-pt} v\, dt = A/m(1 - e^{-ph}). \qquad (35)$$

Now the r.h.s. of (35) satisfies the condition at (3), so by (2) the velocity of m is

$$v = \frac{(A/m)}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}\, dz/z(1 - e^{-zh}) \qquad (36)$$

$$= (A/m)\left\{\frac{t}{h} + \left[\frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2\pi nt/h}{n}\right]\right\}. \qquad (37)$$

It is of interest to consider the type of function represented by formula (37). Expanding (34), we obtain

$$v \supset (A/m)[1 + e^{-ph} + e^{-2ph} + \ldots]. \qquad (38)$$

The first term of (38) represents the Heaviside unit function having an amplitude A/m; the second term represents the same function commencing at $t=h$, and so on. The superimposition of all the unit functions results in an unending staircase, each stair having a height (A/m) and length h. If we draw a straight line from $t=0$ through the lower corners of the stairs, its equation is

$$v_1 = (A/m)\frac{t}{h}. \qquad (39)$$

* This means that there is a time derivative on the l.h.s. of the differential equation, the driving force or disturbance being represented on the r.h.s. by (32).

Removing that portion of the stairs above this line and plotting it above the t axis, we have a saw-tooth wave form. It is clear from (37), (39) that the Fourier expansion of this wave form is given by the two terms in square brackets in (37). The mean value of the saw-tooth function is $A/2m$.

8. *Problems.*

Finally, in the pious hope that they may be of interest, we give two practical problems.

(i) In telegraphic communication between ships at sea, the distress signal S O S (save our souls) is repeated in the Morse code, when this is unfortunately necessary. The letters S and O are represented by three dots and three dashes, respectively. The time occupied by a dash equals that for three dots ; the interval between two elements of a letter is that for a dot, whilst that between two letters is equal in length to three dots. What is the O.F. of the distress signal, the length of a dot being h? If the signal is repeated indefinitely, the interval between consecutive S's being equal in length to two dashes, what is the O.F. and the Fourier expansion of the rythmic signal?

(ii) A note at the top end of a pianoforte is repeated at equal intervals h. The duration of the sound during each interval is h_1, the damper being on the strings for a time $h-h_1$. If overtones are negligible and the sound wave has the form $e^{-at}\cos\omega t$ for an isolated note, what is the Fourier expansion of the repeated note? The simplest way of repeating the note is to use one finger on each hand and play from the wrists. This problem should be useful in developing pianoforte technique!

APPENDIX.

In technical applications (16) may take the form

$$a_0\frac{d^{n-1}v}{dt^{n-1}}+a_1\frac{d^{n-2}v}{dt^{n-2}}+\ldots+a_{n-1}v+a_n\int_0^t v\,dt=\xi(t). \quad\text{.........(40)}$$

Multiplying both sides of (40) by pe^{-pt} and proceeding as in § 3, provided the system is quiescent and no energy is stored therein at $t=0$, *i.e.* $\phi_3(p)=0$, we find that

$$\phi_2(p)=a_0p^{n-1}+a_1p^{n-2}+\ldots+a_{n-1}+\frac{a_n}{p}. \quad\text{............(41)}$$

When the system is not quiescent at $t=0$, we may put

$$\int_0^t v\,dt=y, \quad v=dy/dt$$

and (40) becomes

$$a_0\frac{dy^n}{dt^n}+a_1\frac{dy^{n-1}}{dt^{n-1}}+\ldots+a_{n-1}\frac{dy}{dt}+a_ny=\xi(t). \quad\text{............(42)}$$

Then $\phi_3(p)$, the initial conditions function, is obtained from the array (17).

Having found y as shown in § 3, v is got by differentiation. Since $\phi_2(z)$ and $\phi_3(z)$ in (14) are polynomials, if $\phi_1(z)$ does not introduce branch points, the contour integral for y may be differentiated prior to evaluation. This is treated later.

Example. As a simple illustration, consider the case of a mass m attached to one end of a massless helical spring whose other end is fixed. The mass is free to move along the axis of the spring in a horizontal plane. The stiffness or force per unit change in length of the spring is s, whilst the resistance to motion is rv, v being the velocity of m and r a constant. The system is quiescent and at $t=0$ a force represented by $\xi(t)$ is applied axially to m. What is the velocity, v, at any subsequent time?

1°. The equation of motion is

$$m\frac{dv}{dt}+rv+s\int_0^t v\,dt=\xi(t). \qquad \text{......(43)}$$

The operational form of (43) is

$$\left(mp+r+\frac{s}{p}\right)v=\phi_1(p), \qquad \text{......(44)}$$

so

$$v=p\phi_1(p)/(mp^2+rp+s) \qquad \text{......(45)}$$

or

$$v=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{zt}\,\phi_1(z)\,dz/(mz^2+rz+s), \qquad \text{......(46)}$$

which is evaluated in the usual way.

2°. Writing $v=\dfrac{dy}{dt}$, (43) leads to

$$m\frac{d^2y}{dt^2}+r\frac{dy}{dt}+sy=\xi(t). \qquad \text{......(47)}$$

If at $t=0$ the system is at rest but the spring is compressed an amount y_0 (motion just having ceased momentarily), by (17)

$$\phi_3(p)=(mp^2+rp)y_0. \qquad \text{......(48)}$$

Now

$$\phi_2(p)=mp^2+rp+s, \qquad \text{......(49)}$$

so by (13), (14)

$$y=f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{zt}\left[\frac{\phi_1(z)+(mz^2+rz)y_0}{mz^2+rz+s}\right]\frac{dz}{z}, \qquad \text{......(50)}$$

this being evaluated in the usual way, after which $v=f'(t)$ is found.

If the integrand of (50) is such that the integral round an infinite semicircle on the left of $c\pm i\infty$ is zero, the contour $[-\infty, (0+), -\infty]$ —drawn so that all singularities lie on its left—is equivalent to $c\pm i\infty$. $\phi_2(z)$ and $\phi_3(z)$ are polynomials, but $\phi_1(z)$ may introduce branch points, although in applications this is unlikely. Consequently if all the singularities are poles in the finite part of the z plane, $c\pm i\infty$ may be replaced by a finite circle enclosing them. Under this condition (50) may be differentiated prior to evaluation, since the resulting integral is convergent. N. W. McL.

A GRAPHICAL METHOD OF SOLVING TARTAGLIAN MEASURING PUZZLES.*

BY M. C. K. TWEEDIE.

1.0. A man has three vessels, whose capacities are 3, 5 and 8 pints respectively. The largest is full of water. He desires to divide this water into two equal parts by using these vessels only. What are the simplest ways of doing this?

1.1. *Graphical solution.*

Call the vessels X, Y and Z respectively, and the amounts of water they contain x, y and z respectively.

Then $x+y+z=8$, $0 \leqslant x \leqslant 3$, $0 \leqslant y \leqslant 5$, $0 \leqslant z \leqslant 8$.

Initially, $x=0=y$, $z=8$; finally, $x=0$, $y=4=z$.

Now x, y and z can be taken as the trilinear coordinates of a point with respect to an equilateral triangle of altitude 8. Because of the above restrictions, all the possible positions of the point lie between the pairs of lines $x=0$, $x=3$; $y=0$, $y=5$; $z=0$, $z=8$; or on one or more of these lines. The set of parallel lines $x=0, 1, 2, 3$ corresponds to the different possible amounts of water in X, and the lines $y=0, 1, \ldots, 5$ correspond to the possible amounts in Y. The lines for Z are a set of lines passing through the intersections of the lines for X and Y, the quantities they represent being determined by the equation $x+y+z=8$.

The diagram, then, is

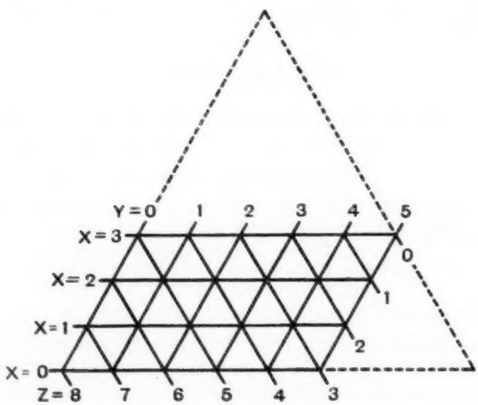


FIG. 1.

* "It is the general opinion that puzzles of this class can only be solved by trial, but I think formulae can be constructed for the solution generally of certain related cases. It is a practically unexplored field for investigation."—Dudeney, *Amusements in Mathematics* (1917), p. 109. I owe this reference to Professor Neville.

The initial state is represented by (0, 0, 8) and the final state by (0, 4, 4).

1.2. Now when water is transferred from one vessel to another the amount of water in the third remains unchanged, and therefore the effect of a single transfer is that the representative point travels along one of the lines drawn, within the limits of the diagram. Further, when a transfer is made, one vessel must be completely emptied or the other completely filled, so that the result of a single transfer is represented by a point on the boundary line. Hence a single transfer is represented on the diagram by one of the lines drawn, extending from one boundary point to another.

1.3. The problem can be solved as follows :

On the diagram first mark with 1 the points that can be reached from (0, 0, 8) in one move. The points, excepting the initial point, that can be reached by a second move mark with 2. Mark with 3 the points that can be reached from these by a third move, excepting those already marked. Continuing thus we can mark against each point the minimum number of moves in which it can be reached. This process takes about half a minute. The essential part of the diagram is then

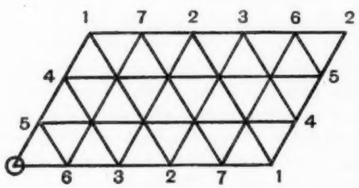


Fig. 2.

Note that every boundary point can be reached, and, in particular, the required final point, (0, 4, 4), can be reached in 7 moves. Examination of the moves leading to it reveals that there is only one shortest route. There is, however, a route involving 8 moves, in which (3, 1, 4) is the penultimate point. These two are the solutions involving the least number of transfers. The diagrams (Fig. 3) show the routes

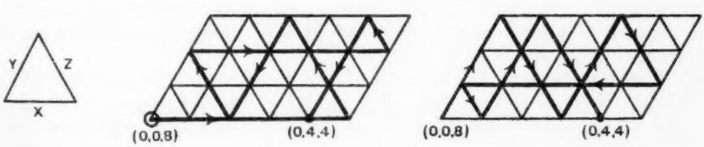


Fig. 3.

and the solutions are

x	y	z	
0	0	8	
0	5	3	
3	2	3	
0	2	6	
2	0	6	
2	5	1	
3	4	1	
0	4	4	(7 transfers)

x	y	z	
0	0	8	
3	0	5	
0	3	5	
3	3	2	
1	5	2	
1	0	7	
0	1	7	
3	1	4	
0	4	4	(8 transfers)

1.4. Observe that the corner remotest from the initial point does not enter into either solution, *i.e.* Z is never completely emptied. The reason for this is that the points immediately accessible from this corner can be reached in one move from the initial point, and consequently the moves leading to the corner are wasted.

After the first move the representative point never travels along a boundary line, since if it did the move would finish in a corner and thus it and the preceding moves would be wasted. In other words, *if a transfer empties either X or Y, the next transfer must put water into it, and if a transfer fills either of them, the next transfer must remove water from it.* If a move leaves one of these two vessels full and the other empty, the next move consists of pouring from one into the other. These rules really make the diagram unnecessary, but it is nevertheless convenient to use it.

Note that except for the first and last moves one track representing a solution can be reflected into the other through the central point of the diagram. The disposition of the numbers in the diagram showing the minimum number of moves in which each point can be reached from (0, 0, 8) displays central symmetry except for the numbers at the initial point and the corner remotest from it. There is therefore a close relationship between the solution of the problem of how to get from (3, 5, 0) to (3, 1, 4) and the solution of the problem considered above.

1.5. Since all boundary points are accessible from (0, 0, 8), which, being at a corner, is accessible from every point, therefore every boundary point is accessible from every point. In the diagram below (Fig. 4), the number against a point indicates the greatest number of moves necessary to reach any other boundary point from that point.

The numbers at the corners of this diagram are each 7 and are higher than at any of the other points. By choosing (0, 0, 8) as the initial point and (0, 4, 4) as the final one, the originator of the problem made it as hard as possible with the capacities of the vessels and the total amount of water as given.

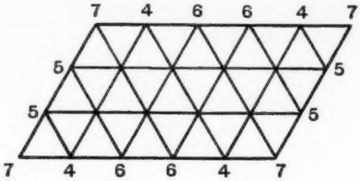


FIG. 4.

As the representative point travels along the shortest track from (0, 0, 8) to (0, 4, 4) or to (3, 1, 4), or from (3, 5, 0) to either of these points, the numbers at the points, in the order taken, are

$$7,\ 7,\ 6,\ 6,\ 5,\ 5,\ 4,\ 4.$$

The numbers are arranged symmetrically about the lines $x = 1\frac{1}{2}$ and $y = 2\frac{1}{2}$.

2.0. A diagram of a similar character is obtained when the capacities of the vessels are a, b and $(a+b)$, and the total amount of water $(a+b)$. If a and b have no common factor all boundary points are accessible from every point, and the properties of the diagram used for the above problem hold, with slight modification, in this more general diagram.

Since only one of a, b and $(a+b)$ is an even number, there is only one line through the centre of the diagram. The points at the ends of this line can be reached in a minimum of $(a+b-1)$ moves from either of the corners $(a, b, 0)$ and $(0, 0, a+b)$, and this is the greatest number necessary to reach any boundary point from any other boundary point. One of these points is also $(a+b-1)$ moves from one of the other corners, and the other is the same number from the fourth corner. There are also two other points which are $(a+b-1)$ moves from the latter two corners.

3.0. In the general case in which the capacities of the three vessels and the initial distribution of the water are not subject to the same relation as in the case considered above, a diagram can be constructed on the same general principles, but it may be a parallelogram, pentagon, hexagon, trapezium or triangle according to circumstances. The process of § 1.3 will give the simplest solutions. The diagram is needed if the problem is to be solved completely reasonably quickly.

3.1. *Problem.*

A man has three jars, whose capacities are 5, 6 and 10 pints. The 5-pint jar is empty and the others contain 6 pints each. He wishes to get 3 pints in one jar, 4 in another and 5 in the other. What are the simplest ways of getting this distribution?

Solution.

The diagram labelled as in § 1.1 is

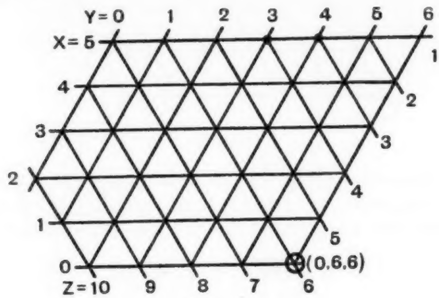


FIG. 5.

and labelled by the method of § 1.3 is

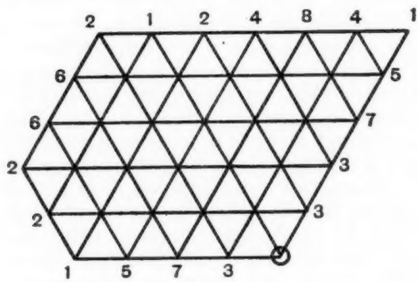


FIG. 6.

The six points (5, 3, 4), (5, 4, 3), (4, 3, 5), (4, 5, 3), (3, 4, 5), (3, 5, 4) each satisfy the conditions required in the final distribution, but only the first two are accessible from (0, 6, 6), since they are the only ones which lie on the boundary of the diagram. The second diagram above (Fig. 6) shows that (5, 3, 4) can be reached in the smaller number of moves, namely, 4. M. C. K. TWEEDIE.

1278. A good example of an inclined plane is a horse pulling a cart up a slope, the horse being the force and the cart the weight.—*Interesting experiments in applied mechanics with Meccano.* [Per Mr. G. H. Grattan-Guinness.]

1279. His respect for Mr. Reeves was rather like the square root of minus one—difficult to perceive and a minus quantity when found.—Richard Aldington, *Seven against Reeves.* [Per Mr. W. T. G. Parker.]

1280. A block of wood . . . It has been trisected into nine parts.—William Seabrook, in *The Reader's Digest*, August, 1938, p. 29. [Per Mr. E. S. Pondiczery.]

MATHEMATICAL NOTES.

1377. *On Notes* 1310, 1348, 1349.

The following direct proof of the identity involved may be of interest; depending, as it does, only on the division process and the well-known property of polygonal numbers:

$$\begin{bmatrix}1\\r\end{bmatrix}+\begin{bmatrix}2\\r\end{bmatrix}+\begin{bmatrix}3\\r\end{bmatrix}+\dots+\begin{bmatrix}s\\r\end{bmatrix}=\begin{bmatrix}s\\r+1\end{bmatrix}. \qquad \dots\dots\dots\dots\dots(1)$$

The identity to be proved is

$$(\alpha+\beta)^n \equiv (\alpha^n+\beta^n)+\begin{bmatrix}n\\1\end{bmatrix}(\alpha^{n-1}+\beta^{n-1})\frac{\alpha\beta}{\alpha+\beta}$$
$$+\begin{bmatrix}n\\2\end{bmatrix}(\alpha^{n-2}+\beta^{n-2})\left(\frac{\alpha\beta}{\alpha+\beta}\right)^2 \dots \text{ to } n \text{ terms.}$$

Putting $x=\beta/(\alpha+\beta)$, so that $\alpha/(\alpha+\beta)=1-x$, this is, after dividing by $(\alpha+\beta)^n$ and rearranging:

$$(1-x)^n\left\{1+\begin{bmatrix}n\\1\end{bmatrix}x+\begin{bmatrix}n\\2\end{bmatrix}x^2+\dots+\begin{bmatrix}n\\n-1\end{bmatrix}x^{n-1}\right\}$$
$$\equiv 1-x^n-\begin{bmatrix}n\\1\end{bmatrix}x^n(1-x)-\begin{bmatrix}n\\2\end{bmatrix}x^n(1-x)^2\dots-\begin{bmatrix}n\\n-1\end{bmatrix}x^n(1-x)^{n-1}.$$

This we may readily see to be true by dividing the second member by $(1-x)$, n times, and using (1). The successive quotients are clearly

$$1+x+x^2+x^3+\dots+x^{n-1}-\begin{bmatrix}n\\1\end{bmatrix}x^n-\begin{bmatrix}n\\2\end{bmatrix}x^n(1-x)$$
$$-\dots-\begin{bmatrix}n\\n-1\end{bmatrix}x^n(1-x)^{n-2},$$

$$1+2x+3x^2+4x^3+\dots+nx^{n-1}-\begin{bmatrix}n\\2\end{bmatrix}x^n-\dots-\begin{bmatrix}n\\n-1\end{bmatrix}x^n(1-x)^{n-3},$$

$$1+3x+6x^2+10x^3+\dots+\begin{bmatrix}n\\2\end{bmatrix}x^{n-1}-\begin{bmatrix}n\\3\end{bmatrix}x^n$$
$$-\dots-\begin{bmatrix}n\\n-1\end{bmatrix}x^n(1-x)^{n-4}$$

and so on, the last quotient being easily seen to be

$$1+\begin{bmatrix}n\\1\end{bmatrix}x+\begin{bmatrix}n\\2\end{bmatrix}x^2+\dots+\begin{bmatrix}n\\n-1\end{bmatrix}x^{n-1},$$

as is required. C. W. Stokes.

1378. *On Note* 1358.

The method given by H. V. Lowry (*Mathematical Gazette*, XXIII (1939), p. 89) for solving a quartic equation seems to be due to

Heilermann (*Zeitschr. Math. Phys.*, 44 (1898), p. 234). If we write down the condition that the conic

$$S \equiv a_0x^2 + 2a_1xy + a_2y^2 + 2a_2x + 2a_3y + a_4 = 0$$

should pass through the point $(t^2, 2t)$, whose locus is the parabola

$$S' \equiv y^2 - 4x = 0,$$

we obtain the quartic equation in its canonical form

$$a_0t^4 + 4a_1t^3 + 6a_2t^2 + 4a_3t + a_4 = 0\ ;$$

and the condition that $S = a_0\theta S'$ may be a line-pair is the " reducing cubic " in its usual textbook form. Consideration of the common self-conjugate triangle of $S=0$ and $S'=0$ gives the familiar theorems concerning the reality of the roots of the quartic equation, etc.

H. SIMPSON.

1379. *Correspondences in projective geometry.*

The following seems a reasonably easy proof of the algebraic one-to-one correspondence theorem (J. A. Todd, *Mathematical Gazette*, XXIII (1939), p. 58), involving only an elementary knowledge of the theory of algebraic plane curves.

Suppose that $f(x, y)$ is a non-factorisable polynomial of degree n such that $f=0$ gives only one distinct value of x and *vice versa.* Then f, $\partial f/\partial x$ have no factor in common and therefore the curves $f=0$, $\partial f/\partial x=0$ meet in only a finite number of points. Hence there are only a finite number of double points on $f=0$ or of points at which the tangent is parallel to $y=0$. Therefore there are an infinite number of values of x for which all the finite values of y given by $f=0$ are distinct. But there is only one such value of y. Hence the polynomials in x which are the coefficients of y^n, y^{n-1}, ..., y^2 are zero for an infinite number of values of x and are therefore identically zero. A similar argument applies to x^n, x^{n-1}, ..., x^2 and the theorem follows.

H. SIMPSON.

1380. *Approximations to π.*

Two approximations which are, I believe, nearer to π than any other fractions not having much greater numbers for numerator and denominator :

(A) $\dfrac{104348}{33215} = 3{\cdot}141592653 \mid 92142 \ldots .$

(B) $\dfrac{833719}{265381} = 3{\cdot}14159265358 \mid 108 \ldots .$

B is 38 times as good as A : that is, its error is $\frac{1}{38}$ of A's.

To remember A : $\dfrac{\text{Calculator will get fair accuracy}}{\text{but not to } \pi \text{ exact}}$.

To remember B : $\dfrac{\text{Dividing top lot through (a nightmare)}}{\text{by number below, you approach } \pi}$.

W. HOPE-JONES.

1381. *The triple vector product formulae.*

The following proofs of these well-known formulae seem to be simple, and they satisfy all the needs of a pure vector formalism. I do not know whether they are new or not.

1. *The scalar product* $(\mathbf{a}[\mathbf{bc}])$. It is obvious that

$$0 = (\mathbf{a}+\mathbf{b}\,.\,[\mathbf{a}+\mathbf{b}\,.\,\mathbf{c}])$$
$$= (\mathbf{a}[\mathbf{bc}]) + (\mathbf{b}[\mathbf{ac}]),$$

whence $$(\mathbf{a}[\mathbf{bc}]) = -(\mathbf{b}[\mathbf{ac}]) = (\mathbf{b}[\mathbf{ca}]).$$

2. *The vector product* $[\mathbf{a}[\mathbf{bc}]]$. Realising that a vector perpendicular to a vector $\mathbf{b}$, in the plane of $\mathbf{b}$ and a second vector $\mathbf{c}$, can always be expressed in the form $\alpha\{(\mathbf{bc})\,\mathbf{b} - (\mathbf{bb})\,\mathbf{c}\}$ we can write

$$[\mathbf{b}[\mathbf{bc}]] = \lambda\{(\mathbf{bc})\,\mathbf{b} - (\mathbf{bb})\,\mathbf{c}\}$$

and scalar multiplication by $\mathbf{c}$ gives

$$(\mathbf{c}[\mathbf{b}[\mathbf{bc}]]) = -([\mathbf{bc}][\mathbf{bc}]) = -\lambda\{(\mathbf{bb})\,(\mathbf{cc}) - (\mathbf{bc})^2\},$$

so that obviously $\lambda = 1$.

Writing now $\mathbf{a}$ in the form $\lambda_1\mathbf{b} + \lambda_2\mathbf{c} + \lambda_3[\mathbf{bc}]$, where the scalar coefficients λ_1, λ_2 are given by

$$(\mathbf{ab}) = \lambda_1(\mathbf{bb}) + \lambda_2(\mathbf{bc}),$$
$$(\mathbf{ac}) = \lambda_1(\mathbf{bc}) + \lambda_2(\mathbf{cc}),$$

we see at once that

$$[\mathbf{a}[\mathbf{bc}]] = \lambda_1[\mathbf{b}[\mathbf{bc}]] + \lambda_2[\mathbf{c}[\mathbf{bc}]]$$
$$= \lambda_1\{(\mathbf{bc})\,\mathbf{b} - (\mathbf{bb})\mathbf{c}\} + \lambda_2\{(\mathbf{cc})\,\mathbf{b} - (\mathbf{bc})\,\mathbf{c}\}$$
$$= (\mathbf{ca})\,\mathbf{b} - (\mathbf{ab})\mathbf{c}.$$

G. H. Livens.

1382. *The multiplication of determinants and vector analysis.*

The usual law of multiplication for third order determinants can be written in vector notation as

[1] $$[\mathbf{a}\,.\,\mathbf{b}\wedge\mathbf{c}][\mathbf{a}'\,.\,\mathbf{b}'\wedge\mathbf{c}'] = \begin{vmatrix} \mathbf{a}\,.\,\mathbf{a}' & \mathbf{a}\,.\,\mathbf{b}' & \mathbf{a}\,.\,\mathbf{c}' \\ \mathbf{b}\,.\,\mathbf{a}' & \mathbf{b}\,.\,\mathbf{b}' & \mathbf{b}\,.\,\mathbf{c}' \\ \mathbf{c}\,.\,\mathbf{a}' & \mathbf{c}\,.\,\mathbf{b}' & \mathbf{c}\,.\,\mathbf{c}' \end{vmatrix}.$$

In particular, this gives

[2] $$[\mathbf{a}\,.\,\mathbf{b}\wedge\mathbf{c}]^2 = \begin{vmatrix} \mathbf{a}^2 & \mathbf{a}\,.\,\mathbf{b} & \mathbf{a}\,.\,\mathbf{c} \\ \mathbf{a}\,.\,\mathbf{b} & \mathbf{b}^2 & \mathbf{b}\,.\,\mathbf{c} \\ \mathbf{a}\,.\,\mathbf{c} & \mathbf{b}\,.\,\mathbf{c} & \mathbf{c}^2 \end{vmatrix}.$$

If, now, we put $\mathbf{a} = \mathbf{b}\wedge\mathbf{c}$ in [2], we obtain

[3] $$[\mathbf{b}\wedge\mathbf{c}]^4 = \begin{vmatrix} [\mathbf{b}\wedge\mathbf{c}]^2 & 0 & 0 \\ 0 & \mathbf{b}^2 & \mathbf{b}\,.\,\mathbf{c} \\ 0 & \mathbf{b}\,.\,\mathbf{c} & \mathbf{c}^2 \end{vmatrix} = (\mathbf{b}\wedge\mathbf{c})^2\{\mathbf{b}^2\mathbf{c}^2 - (\mathbf{b}\,.\,\mathbf{c})^2\}.$$

For $\mathbf{b}\wedge\mathbf{c} \neq 0$, this leads to the famous " Lagrange identity ",

[4] $(\mathbf{b}\wedge\mathbf{c})^2=\mathbf{b}^2\mathbf{c}^2-[\mathbf{b}\,.\,\mathbf{c}]^2$,

which is also valid if $\mathbf{b}\wedge\mathbf{c}=0$, as is seen by direct substitution of $\mathbf{b}=0$, or $\mathbf{c}=0$, or $\mathbf{b}=\mu\mathbf{c}$ in [4].

Now [2] and [4] can both be used to evaluate the vector triple product $\mathbf{a}\wedge[\mathbf{b}\wedge\mathbf{c}]$. By the usual arguments that the triple product must lie in the plane of $\mathbf{b}$ and $\mathbf{c}$ and must be perpendicular to $\mathbf{a}$, we arrive at

[5] $\mathbf{a}\wedge[\mathbf{b}\wedge\mathbf{c}]=\lambda[(\mathbf{a}\,.\,\mathbf{c})\mathbf{b}-(\mathbf{a}\,.\,\mathbf{b})\mathbf{c}]$.

To determine the scalar factor λ, take the scalar product of each side with itself, using Lagrange's identity twice for evaluating the left-hand side. This leads at once to

[6] $$\mathbf{a}^2\mathbf{b}^2\mathbf{c}^2-\mathbf{a}^2[\mathbf{b}\,.\,\mathbf{c}]^2-[\mathbf{a}\,.\,\mathbf{b}\wedge\mathbf{c}]^2$$
$$=\lambda^2\{[\mathbf{a}\,.\,\mathbf{c}]^2\mathbf{b}^2+[\mathbf{a}\,.\,\mathbf{b}]^2\mathbf{c}^2-2[\mathbf{a}\,.\,\mathbf{b}][\mathbf{b}\,.\,\mathbf{c}][\mathbf{c}\,.\,\mathbf{a}]\}.$$

But the direct expansion of [2] gives us

[7] $$[\mathbf{a}\,.\,\mathbf{b}\wedge\mathbf{c}]^2$$
$$=\mathbf{a}^2\mathbf{b}^2\mathbf{c}^2+2[\mathbf{a}\,.\,\mathbf{b}][\mathbf{b}\,.\,\mathbf{c}][\mathbf{c}\,.\,\mathbf{a}]-\mathbf{a}^2[\mathbf{b}\,.\,\mathbf{c}]^2-\mathbf{b}^2[\mathbf{a}\,.\,\mathbf{c}]^2-\mathbf{c}^2[\mathbf{a}\,.\,\mathbf{b}]^2.$$

A comparison of [6] and [7] for values of $[\mathbf{a}\,.\,\mathbf{b}\wedge\mathbf{c}]^2$ shows that $\lambda^2=1$, *i.e.* $\lambda=\pm1$, and the value $\lambda=+1$ is determined by specialisation of the vectors, say $\mathbf{a}=\mathbf{b}=\mathbf{i}$, $\mathbf{c}=\mathbf{j}$. (Cf. S. Chapman and E. A. Milne, *Math. Gazette*, XXIII, February 1939, pp. 35-38.)

D. D. Kosambi.

1383. *The vector triple product.*

The formula

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})=(\mathbf{a}\,.\,\mathbf{c})\,\mathbf{b}-(\mathbf{a}\,.\,\mathbf{b})\,\mathbf{c}$$

may be proved for unit vectors, any scalar factors being applied afterwards.

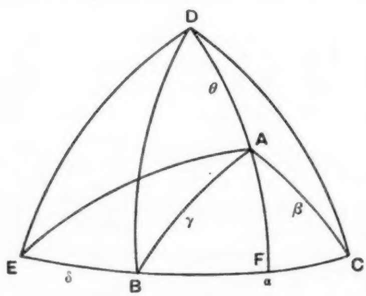


Fig. 1.

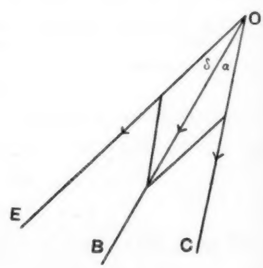


Fig. 2.

Thus if $\mathbf{a}=\overrightarrow{OA}$, $\mathbf{b}=\overrightarrow{OB}$, $\mathbf{c}=\overrightarrow{OC}$, then A, B, C form a spherical triangle with sides α, β, γ. Let D be the pole of BC, and E on BC

the pole of AD in the same sense. Let $\overrightarrow{OD}=\mathbf{d}$, $\overrightarrow{OE}=\mathbf{e}$, $EB=\delta$ and $DA=90^\circ-AEB=\theta$.

Then $\cos\gamma=\sin\delta\sin\theta, \quad \cos\beta=\sin(\alpha+\delta)\sin\theta.$

Hence $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})=\mathbf{a}\wedge(\sin\alpha\,.\,\mathbf{d})=\sin\alpha\sin\theta\,.\,\mathbf{e},$

where $$\mathbf{e}=\frac{\sin(\alpha+\delta)}{\sin\alpha}\cdot\mathbf{b}-\frac{\sin\delta}{\sin\alpha}\cdot\mathbf{c}.$$

Therefore $$\begin{aligned}\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})&=\cos\beta\,.\,\mathbf{b}-\cos\gamma\,.\,\mathbf{c}\\&=(\mathbf{a}\,.\,\mathbf{c})\mathbf{b}-(\mathbf{a}\,.\,\mathbf{b})\mathbf{c}.\end{aligned}$$

H. C. P.

1384. *The vector triple product.*

In this Note a proof of the formula

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})=(\mathbf{a}\,.\,\mathbf{c})\mathbf{b}-(\mathbf{a}\,.\,\mathbf{b})\mathbf{c}$$

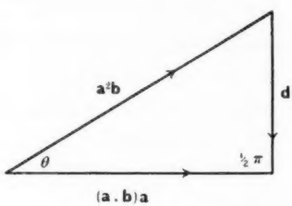
is offered as an alternative to those given in a recent Note in this *Gazette*.

Lemma. $\mathbf{a}\wedge(\mathbf{a}\wedge\mathbf{b})=(\mathbf{a}\,.\,\mathbf{b})\mathbf{a}-\mathbf{a}^2\mathbf{b}.$

$\mathbf{d}$, $=\mathbf{a}\wedge(\mathbf{a}\wedge\mathbf{b})$, is perpendicular to $\mathbf{a}$ in the plane of $\mathbf{a}$, $\mathbf{b}$ and, by following the corkscrews round, we can see that the angle between the positive directions of $\mathbf{d}$ and $\mathbf{b}$ is obtuse.

Further $|\,\mathbf{d}\,|=\mathbf{a}^2\,|\,\mathbf{b}\,|\sin\theta=\sqrt{[(\mathbf{a}^2\mathbf{b})^2-\{(\mathbf{a}\,.\,\mathbf{b})\mathbf{a}\}^2]}.$

Thus we can construct the vector diagram:



This proves the lemma.

Next, as in the Note quoted,

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})=\lambda\{(\mathbf{a}\,.\,\mathbf{c})\mathbf{b}-(\mathbf{a}\,.\,\mathbf{b})\mathbf{c}\}$$

and therefore

$$\begin{aligned}\lambda\{(\mathbf{a}\,.\,\mathbf{c})\mathbf{b}-(\mathbf{a}\,.\,\mathbf{b})\mathbf{c}\}\,.\,\mathbf{b}&=\{\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})\}\,.\,\mathbf{b}\\&=\mathbf{a}\,.\,\{(\mathbf{b}\wedge\mathbf{c})\wedge\mathbf{b}\}\\&=\mathbf{a}\,.\,\{\mathbf{b}^2\mathbf{c}-(\mathbf{b}\,.\,\mathbf{c})\mathbf{b}\}.\end{aligned}$$

Whence $\lambda=1$.

T. G. ROOM.

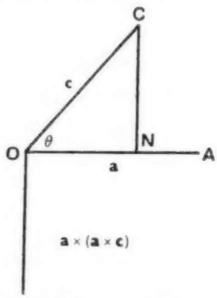
1385. *The triple vector product.*

The following proof is a modification of one which I have used for some years, of the identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\,.\,\mathbf{c})\mathbf{b}-(\mathbf{a}\,.\,\mathbf{b})\mathbf{c}. \quad \text{......................(1)}$$

In it the properties of the scalar triple product $\mathbf{a}\,.\,(\mathbf{b}\times\mathbf{c})$ are assumed. I shall take **a**, **b**, **c** as unit vectors, for if it is proved for them the general case follows; this makes a slight simplification but may easily be avoided if required.

If then **a**, **b**, **c** are unit vectors the special case of the formula when $\mathbf{a}=\mathbf{b}$ is obvious from definition, for $\mathbf{a}\times(\mathbf{a}\times\mathbf{c})$ is a vector in



the plane of **a** and **c** of modulus $\sin\theta$ (in the figure) and is perpendicular to **a**. It is therefore represented by $\overrightarrow{CN}$ or $\overrightarrow{CO}+\overrightarrow{ON}$ or is

$$-\mathbf{c}+\mathbf{a}\cos\theta=(\mathbf{a}\,.\,\mathbf{c})\mathbf{a}-\mathbf{c}. \quad \text{.........................(2)}$$

In the general case suppose **a**, **b**, **c** non-coplanar, then we can write

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=A\mathbf{a}+B\mathbf{b}+C\mathbf{c}. \quad \text{.......................(3)}$$

Take the scalar product by $\mathbf{b}\times\mathbf{c}$, $\mathbf{c}\times\mathbf{a}$, $\mathbf{a}\times\mathbf{b}$ in turn. The first gives

$$0=A\mathbf{a}\,.\,(\mathbf{b}\times\mathbf{c}).$$

Thus $\qquad A=0.$

The second gives on the left

$$\begin{aligned}&(\mathbf{c}\times\mathbf{a})\,.\,\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\}\\ &=(\mathbf{b}\times\mathbf{c})\,.\,\{(\mathbf{c}\times\mathbf{a})\times\mathbf{a}\}\\ &=(\mathbf{b}\times\mathbf{c})\,.\,\{(\mathbf{a}\,.\,\mathbf{c})\mathbf{a}-\mathbf{c}\} \quad \text{by (2)}\\ &=(\mathbf{a}\,.\,\mathbf{c})\{\mathbf{a}\,.\,(\mathbf{b}\times\mathbf{c})\},\end{aligned}$$

since $\mathbf{c}\,.\,(\mathbf{b}\times\mathbf{c})=0$.

On the right we have only $B\mathbf{b}\,.\,(\mathbf{c}\times\mathbf{a})$.

Thus $\qquad B=\mathbf{a}\,.\,\mathbf{c}.$

Similarly we get C or use the common method of taking the scalar product of (2) by **a**, giving

$$B\mathbf{a}\,.\,\mathbf{b}+C\mathbf{a}\,.\,\mathbf{c}=0.$$

If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar we can use the common method to show that $A=0$ and the rest follows as before.

May I add some remarks on the use of vectors in general?

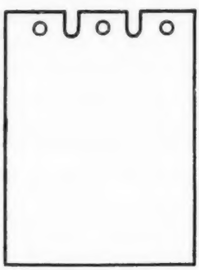
First, why is there such a divergence in the notation for scalar and vector products? Surely it is time for the general adoption of Gibbs's dot and cross notation. It is extraordinarily convenient for writing, printing and speaking.

Secondly, the difficulty I find with students is that they do not use sufficiently distinctive letters for vectors. Greek letters are convenient, but we often want them for scalars as well. It is more satisfactory to use heavy small printed letters, writing them in the same way as they are printed in this note. Also in place of the heavy block capitals in printed books we may use ornamental script capitals in writing.

Thirdly, in such expressions as occur in this paper students should not be sparing of the brackets, even if the expressions are unambiguous if the brackets are omitted. R. J. A. BARNARD.

1386. *A sorting problem.*

In some businesses where card indexes are used, the following method of sorting the cards after use is employed. All the cards have holes punched in the same positions, and some of the cards have some of the holes slotted. The cards are placed to form a pack, a peg is passed through the first hole and the pack shaken so that all cards with the first hole slotted fall out without disturbing their order. These are placed at the back. The operation is repeated for the second, third and other holes; at the end the whole pack is found to be arranged in its correct order.



The problem is to find the number of holes required and the system of slotting for a pack of n cards.

There are several empirical ways of finding a solution for any given value of n, and the solution is not always unique. A general solution shows an unexpected connection with various problems depending on expressing a number in the scale of 2, for example, the fact that any weight can be weighed by the use of weights (one

for each), 1 lb., 2 lb., 4 lb., etc. For convenience, let the cards in the pack be numbered in sequence 0, 1, 2, 3, 4, ... , and the holes 0, 1, 2, ... , then for the card numbered N, if $N = 2^a + 2^b + 2^c$, etc., the holes numbered a, b, c, etc., are slotted.

The table given here is used in a game of telling what number a person thinks of, and it enables numbers to be expressed immediately as the sum of powers of 2, and so enables the slotting to be quickly determined.

A	*B*	*C*	*D*
1	2	4	8
3	3	5	9
5	6	6	10
7	7	7	11
9	10	12	12
11	11	13	13
13	14	14	14
15	15	15	15

For example, if a number occurs in columns A, C, D, it is the sum of the numbers at the head of those columns, namely, it is $1+4+8$. Thus $13 = 2^0 + 2^2 + 2^3$. All cards bearing numbers in column A have the first hole slotted, all bearing numbers in column B have the second hole slotted, and so on.

This is the solution; so the card numbered 13 would have the first, third and fourth holes slotted.

The working can be seen by using the following notation: | 1356 | represents the sub-pack of cards numbered 1, 3, 5 and 6, but in no particular order; | 16 | 35 | shows that the sub-pack | 35 | follows the sub-pack | 16 |. Then for 16 cards numbered 0 to 15,

| 0 2 4 6 8 10 12 14 || 1 3 5 7 9 11 13 15 |

represents the order after the first operation;

| 0 4 8 12 | 1 5 9 13 || 2 6 10 14 | 3 7 11 15 |

represents the order after the second operation;

| 0 8 | 1 9 | 2 10 | 3 11 || 4 12 | 5 13 | 6 14 | 7 15 |

represents the order after the third operation;

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 || 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |

represents the order after the fourth operation, the last 8 cards in each case being those removed by shaking during the operation.

For a pack of 2^m cards (or any number between $2^{m-1}+1$ and 2^m inclusive), m holes are required and 2^{m-1} cards are removed at each operation. F. C. B.

1387. *A simple proof of Feuerbach's Theorem.*

In the figure, O is the circumcentre of the triangle ABC, OD the perpendicular upon BC bisecting it at D and produced to meet the circumcircle at V, I the incentre situated in AV such that

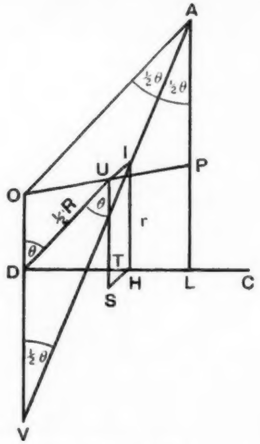


FIG. 1.

$VI = VC$, $IH\,(=r)$ and APL perpendiculars upon BC, P the orthocentre, U the nine-point centre situated at the middle point of OP; $DU\,(=\tfrac{1}{2}R)$ is parallel to $OA\,(=R)$; VA bisects the angle OAP, which equals $C - B = \theta$ (say).

Then
$$\begin{aligned} DH\,.\,HL &= VI \sin \tfrac{1}{2}\theta \,.\, IA \sin \tfrac{1}{2}\theta \\ &= VC\,.\,IA\,.\sin^2 \tfrac{1}{2}\theta \\ &= 2R \sin \tfrac{1}{2}A\,.\,r \operatorname{cosec} \tfrac{1}{2}A\,.\sin^2 \tfrac{1}{2}\theta \\ &= Rr(1 - \cos\theta). \end{aligned}$$

Now complete the parallelogram $UIHS$, so that US bisects DL at right angles at T.

Thus
$$\begin{aligned} Rr - Rr\cos\theta &= DT^2 - TH^2 \\ &= DS^2 - SH^2 \\ &= (\tfrac{1}{4}R^2 + r^2 - Rr\cos\theta) - UI^2. \end{aligned}$$

Thus
$$UI^2 = \tfrac{1}{4}R^2 - Rr + r^2 = (\tfrac{1}{2}R - r)^2.$$

As the distance between the centres of the incircle and the nine-point circle is equal to the difference of their radii, it follows that *these circles touch internally*.

Similarly it is proved that *the nine-point circle touches each of the three escribed circles externally.*

Otherwise. Let the perpendicular UT from U (the nine-point centre) upon the side BC, bisecting DL at T, meet the nine-point circle at X on the side of BC remote from A, and let XH meet the nine-point circle again at Y.

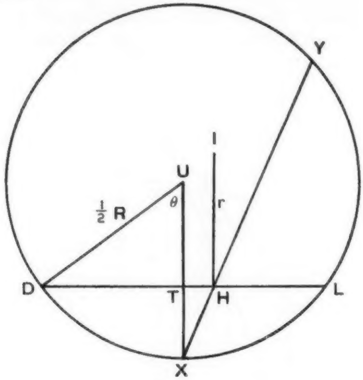


FIG. 2.

Then $\quad XH \,.\, HY = DH \,.\, HL = 2Rr \sin^2 \tfrac{1}{2}\theta$

and $\quad XH \,.\, XY = XD^2 \quad = R^2 \sin^2 \tfrac{1}{2}\theta.$

Thus $\quad HY/XY = r/\tfrac{1}{2}R \quad = HI/XU.$

Hence UIY is a straight line, $IY = r$, and $UI = \tfrac{1}{2}R - r$.

Thus the in-circle touches the nine-point circle internally at Y.

In the same way, if H_1 is taken so that D is the middle point of HH_1, it is proved that the escribed circle which touches the side BC at H_1 also touches the nine-point circle externally at Y_1 in XH_1.

W. J. D.

1388. *The addition formula.*

Note 1320 (October, 1938) reminds me of a proof which I used to favour and which can be applied to angles of any magnitude.

First, if A and B can be the angles of a triangle, construct the triangle ABC of which A and B are two angles.

Then $\quad c = a \cos B + b \cos A,$

$$2R \sin C = 2R \sin A \cos B + 2R \sin B \cos A,$$

and so $\quad \sin C = \sin(A + B)$

$$= \sin A \cos B + \sin B \cos A. \quad \text{......................(i)}$$

For other angles, say P and Q, draw a straight line AB, and draw at A in an anti-clockwise direction an angle BAL equal to P, and at B an angle ABM, in a clockwise direction, equal to Q.

Let AL and BM meet in C.

Example 1. Let P be a positive acute angle and Q a smaller negative angle.

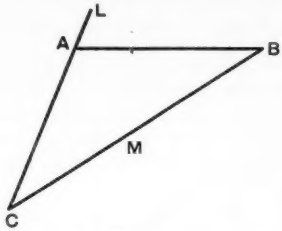


FIG. 1.

In the triangle ABC we have

$$A = 180° - P,$$
$$B = -Q,$$
$$C = P - B = P + Q.$$

Substituting in (i) we can have

either $\sin(P - B) = \sin(180° - P)\cos B + \cos(180° - P)\sin B$,

that is, $\sin(P - B) = \sin P \cos B - \cos P \sin B$,

the difference formula ;

or $\sin(P + Q) = \sin(180° - P)\cos(-Q) + \cos(180° - P)\sin(-Q)$,

that is, $\sin(P + Q) = \sin P \cos Q + (-\cos P)(-\sin Q)$,

a case of the addition formula.

Example 2. Take P so that $270° > P > 180°$ and Q an acute angle.

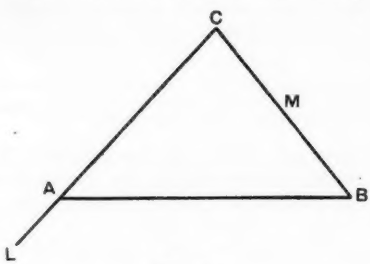


FIG. 2.

Now in the triangle ABC,

$$A = P - 180°,$$
$$B = Q,$$
$$C = (180° - A - B) = (360° - P - Q).$$

Substituting in (i),

$$\sin(360^\circ - P - Q) = \sin(P - 180^\circ)\cos Q + \cos(P - 180^\circ)\sin Q,$$
$$-\sin(P+Q) = -\sin P\cos Q - \cos P\sin Q.$$

All signs required can be determined by inspecting the graph (Fig. 3), which is the sine graph when O is the origin and the cosine

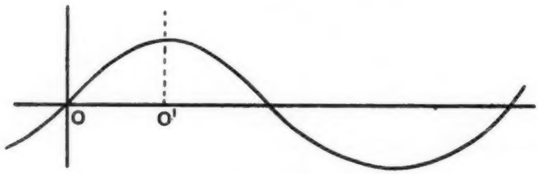


FIG. 3.

graph when O' is the origin.

N.B. The difference formula is obtained by drawing both angles in the same direction, either clockwise or anti-clockwise. F. C. B.

1389. *On Note* 1325 : *solution to problem* 1.

In the continued fraction

$$\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots,$$

where the a's are positive quantities which may be small, and ultimately very small, the usual rules hold for the formation of successive convergents ; also the odd convergents form a descending sequence, the even convergents an ascending sequence ; also we have

$$\frac{p_{2r-1}}{q_{2r-1}} - \frac{p_{2r}}{q_{2r}} = \frac{+1}{q_{2r-1}q_{2r}}, \quad \text{.........................(1)}$$

so that the two sequences never pass each other. Each is a monotonic sequence within a finite range and therefore must tend to a definite limit. *The question is whether the two limits are identical.*

Since $$q_{r+1} = a_{r+1}q_r + q_{r-1}, \quad \text{.........................(2)}$$

q_{r+1} is greater than q_{r-1} but not necessarily greater than q_r. Thus the *even* q's form an ascending sequence *and* the *odd* q's form an ascending sequence.

By (1) it is necessary and sufficient, for identity of limits of the odd and even convergents, that *one* of these sequences should tend to infinity.

By (2) we have

$$\frac{q_3}{q_1}\times\frac{q_5}{q_3}\times\frac{q_7}{q_5}\times\ldots=\left(1+a_3\frac{q_2}{q_1}\right)\left(1+a_5\frac{q_4}{q_3}\right)\left(1+a_7\frac{q_6}{q_5}\right)\ldots,$$

$$\frac{q_4}{q_2}\times\frac{q_6}{q_4}\times\frac{q_8}{q_6}\times\ldots=\left(1+a_4\frac{q_3}{q_2}\right)\left(1+a_6\frac{q_5}{q_4}\right)\left(1+a_8\frac{q_7}{q_5}\right)\ldots.$$

The condition then is that one of the two products on the right should tend to infinity—that is, that one of the two related series should tend to infinity—that is, that the sum of the two should tend to infinity or

$$a_3\frac{q_2}{q_1}+a_4\frac{q_3}{q_2}+a_5\frac{q_4}{q_3}+a_6\frac{q_5}{q_4}+\ldots$$

must tend to infinity.

But $$a_3\frac{q_2}{q_1}+a_4\frac{q_3}{q_2}>2\surd(a_3a_4)\sqrt{\frac{q_3}{q_1}}>2\surd(a_3a_4),$$

$$a_5\frac{q_4}{q_3}+a_6\frac{q_5}{q_4}>2\surd(a_5a_6)\sqrt{\frac{q_5}{q_3}}>2\surd(a_5a_6),$$

etc., etc.

It is therefore sufficient that

$$\surd(a_1a_2)+\surd(a_3a_4)+\surd(a_5a_6)+\ldots$$

should tend to infinity; a condition obviously satisfied in the case proposed, where a_n lies between $1/n$ and $1/(n+1)$; so that the series is comparable with $\Sigma(1/n)$.

The continued fraction is therefore convergent.

Percy J. Heawood.

1390. *A note on conics.*

If through a point O in the plane of the conic $S=0$ a line is drawn to cut the conic in P and Q, and if points A, G, H, R, are located on the line so that OA, OG, OH, OR are respectively the arithmetic, geometric, harmonic means and "root-mean-square" of OP and OQ, and if the line is rotated about O, we find without trouble the following loci:

P and Q,	$S\equiv u_2+u_1+u_0=0.$
H,	$L\equiv 0{\cdot}5u_1+u_0=0.$
A,	$S-L\equiv u_2+0{\cdot}5u_1=0.$
G,	$S-2L\equiv u_2-u_0=0.$
R,	$(u_2-u_1)S+u_1L\equiv u_2(u_2+u_0)-0{\cdot}5u_1^2=0.$

Tangents from O to $S=0$,

$$u_0S-L^2\equiv u_2u_0-0{\cdot}25u_1^2=0.$$

Here are some immediate consequences. The loci of A and G are homothetic to $S=0$. The locus of G has O for centre. The

locus of A has its centre midway between O and the centre of $S=0$. The quartic which is the locus of R has the origin for a double point and for a centre. N. Anning.

1391. *Pythagorean numbers.*

Many years ago I found a note on Pythagorean numbers, from which I compiled the following table. The author did not give any formula from which he had deduced what he wrote, but I attach the rules I followed in compiling the table. After this lapse of time I do not remember either the source or the author.

Squares of the odd numbers	1	2	3	4	5	...... Natural numbers
	2	8	18	32	50	...... Multiply each above by 2, 4, 6, etc.
1	**5**	**13**	**25**	**41**	**61**	...... Add to above 3, 5, 7, etc. respectively
9	**17**	**29**	**45**	**65**	**89**	...... „ „ 12, 16, 20 „ „
25	**37**	**53**	**73**	**97**	**125**	...... „ „ 20, 24, 28 „ „
49	**65**	**85**	**109**	**137**	**169**	...... „ „ 28, 32, 36 „ „
81	**101**	**125**	**153**	**185**	**221**	...... „ „ 36, 40, 44 „ „
121	**145**	**173**	**205**	**241**	**281**	...... „ „ 44, 48, 52 „ „
...	...	...	...	...	...	...... „ „ .... „ „
...	...	...	...	...	...	...... „ „ .... „ „

The lengths of the hypotenuses are represented by the clarendon figures. To determine the other sides of the right-angled triangles, subtract the figures in the vertical and horizontal column and row opposite the number.

For example, if the hypotenuse is 281, then the other sides are $281-50=231$ and $281-121=160$. Therefore the sides of a right-angled triangle may be 281, 231 and 160 units. W. A. Gilmour.

1392. *The division transformation.*

An explicit form of the division transformation can be obtained as follows. Let $f(x)=\sum\limits_0^n a_r x^{n-r}\,(a_0=1)$ and $f_s(x)=\sum\limits_0^s a_r x^{s-r}$; we have

$$f(y)=(y-x)[y^{n-1}+f_1(x)\,y^{n-2}+\ldots+f_{n-1}(x)]+f(x).$$

Hence

$$(y-x)^{-1}=[y^{n-1}+f_1(x)\,y^{n-2}+\ldots+f_{n-1}(x)+f(x)\,(y-x)^{-1}]/f(y),$$

$$(1-xy^{-1})^{-1}=[1+f_1(x)\,y^{-1}+\ldots+f_{n-1}(x)\,y^{-n+1}$$
$$+y^{-n}(1-xy^{-1})^{-1}f(x)][1+a_1y^{-1}+a_2y^{-2}+\ldots+a_ny^{-n}]^{-1}.$$

Expanding in powers of y^{-1} and setting $x^s f(x)=f_{n+s}(x)$ we get

$$1+\Sigma x^r y^{-r}=[1+\Sigma f_r(x)\,y^{-r}][1+\Sigma h_r y^{-r}],$$

where h_r is the total symmetric function of the roots of $f(x)$ of weight r, and on comparing coefficients of y^{-r} we have

$$x^r = f_r(x) + h_1 f_{r-1}(x) + \dots + h_r f_0(x),$$

whence, if $g(x) = \Sigma b_r x^{m-r}$,

$$g(x) = [\sum_{s=n}^{m} \sum_{r=s}^{m} b_{m-r} h_{r-s} x^{s-n}] f(x) + \sum_{s=0}^{n-1} [\sum_{r=s}^{m} b_{m-r} h_{r-s}] f_s(x).$$

Although infinite series are used, the results are algebraic and no question of convergence is involved; this may be seen clearly by taking y^{-1} to be a nilpotent matrix of sufficiently high index.

J. H. M. WEDDERBURN.

1393. *Computation of π by aliquot parts.*

On reading Mr. Hope-Jones's remarks in the February number of the *Gazette* [Vol. XXIII, pp. 13-14] I was led to express π by means of aliquot parts in the following form. This leads to easy calculations, shows rapid convergence, and avoids the division by 113.

$$\pi \doteqdot 3 + 1/7 - 1/7 \times 100 + 1/7 \times 100 \times 9 + 1/7 \times 100 \times 9 \times 30.$$

This gives an error of about 6×10^{-8}, about a quarter of the error of the approximation 355/113. I thought this was close enough and did not pursue the matter further; but on my communicating the result to Mr. Hope-Jones he pointed out that by bringing in the further aliquot part 1/90 the error is further reduced, viz. to about 2×10^{-9}. The following calculation shows the error at various stages.

	3·	To this stage	
($\frac{1}{7}$)	·142 857 142 9 $\div$ 100	error about	$+1{\cdot}3 \times 10^{-3}$
	− *1 428 571 4* $\div$ 9	,, ,,	$-1{\cdot}6 \times 10^{-4}$
	158 730 2 $\div$ 30	,, ,,	$-5{\cdot}4 \times 10^{-6}$
	5 291 0 $\div$ 90	,, ,,	$-6{\cdot}1 \times 10^{-8}$
	58 8	,, ,,	$-2{\cdot}1 \times 10^{-9}$
	3·141 592 651 5		
355/113	3·141 592 920 4	Error about	$+2{\cdot}7 \times 10^{-7}$
π	3·141 592 653 6		

The following are examples of multiplication by this rule: the first is Hope-Jones's.

7·29 × 3	21·870	$\pi^{-1}=$	·3183099
7·29 × $\frac{1}{7}$	1·041 429 $\div$ 100	× 3	·9549297
	− *10 414* $\div$ 9	× $\frac{1}{7}$	·0454728
	1 157 $\div$ 30	−	*·0004547*
	39		505
7·29π	22·902211		17
			1·0000000

For purposes of *division* by π it may be useful to have a similar

expression for π^{-1}. I have not succeeded in finding one quite so simple, but it may be worth while to give the following, which is based on the ratio 113/355 and has an actual error of about $2{\cdot}7\times10^{-8}$, or less than one part in 10 millions of the whole.

$$\frac{113}{355}=\frac{226}{710}=\left(2+\frac{1}{4}+\frac{1}{100}\right)\left(\frac{1}{7}-\frac{1}{10\times7^2}+\frac{1}{10^2\times7^3}-\frac{1}{10^3\times7^4}+\frac{1}{10^4\times7^5}\right).$$

$$\begin{array}{lr}
 & 2 \\
\frac{1}{4} & {\cdot}25 \\
\frac{1}{100} & {\cdot}01 \\
7\,) & \overline{2{\cdot}26} \\
 & {\cdot}322\,857\,143\div70 \\
- & \mathit{4\,612\,245}\div70 \\
 & 65\,889\div70 \\
- & \mathit{941}\div70 \\
 & 13 \\
\hline
 & {\cdot}318\,309\,859 \\
\pi^{-1} & {\cdot}318\,309\,886 \\
\text{Error} & 2{\cdot}7\times10^{-8}
\end{array}
\qquad\qquad
\begin{array}{lr}
\pi= & 3{\cdot}141\,592\,654 \\
\hline
2\pi & 6{\cdot}283\,185\,308 \\
\frac{1}{4}\pi & {\cdot}785\,398\,164 \\
\frac{1}{100}\pi & {\cdot}031\,415\,927 \\
\hline
7\,) & 7{\cdot}099\,999\,399 \\
\hline
 & 1{\cdot}014\,285\,628\div70 \\
- & \mathit{14\,489\,795}\div70 \\
 & 206\,997\div70 \\
- & \mathit{2\,957}\div70 \\
 & 42 \\
\hline
 & {\cdot}999\,999\,915 \\
\pi\times\pi^{-1} & 1{\cdot} \\
\text{Error} & 8{\cdot}5\times10^{-8}
\end{array}$$

G. J. LIDSTONE.

1394. *Notes for lessons on the factors of $\Sigma(b-c)^m$.*

Equipment required :

(*a*) If $f(a)=0$, then $x-a$ is a factor of $f(x)$; and if also $f'(a)=0$, then $(x-a)^2$ is a factor of $f(x)$.

(*b*) $a^2+b^2+c^2-bc-ca-ab=(a+\omega b+\omega^2c)(a+\omega^2b+\omega c)$.

(*c*) Binomial and logarithmic expansions; method of equating coefficients.

Sequence :

1. When m is odd the expression vanishes for $b=c$, $c=a$, $a=b$, hence $(b-c)(c-a)(a-b)$ is a factor; when, further, m is prime, m is also a factor. When m is even, $(b-c)(c-a)(a-b)$ is not a factor.

2. Let $$f(a)=(b-c)^m+(c-a)^m+(a-b)^m.$$
Then $$f(-\omega b-\omega^2c)=(b-c)^m(1+\omega^m+\omega^{2m})$$
and $$f(-\omega^2b-\omega c)=(b-c)^m(1+\omega^{2m}+\omega^m),$$
which are zero when m is of the form $3k\pm1$ but not when m is of the form $3k$.

Also $$f'(-\omega b-\omega^2c)=m(b-c)^{m-1}\omega^{m-1}(\omega^{m-1}-1)$$
and $$f'(-\omega^2b-\omega c)=m(b-c)^{m-1}\omega^{m-1}(1-\omega^{m-1}),$$
which are zero only when m is of the form $3k+1$.

Again,
$$f''(-\omega b-\omega^2c)=m(m-1)(b-c)^{m-2}\omega^{m-2}(\omega^{m-2}+1)$$

and

$$f''(-\omega^2 b-\omega c)=m(m-1)(b-c)^{m-2}\omega^{m-2}(1+\omega^{m-2}),$$

which are never zero.

3. *Summary :*

Form of m	Factors of $\Sigma(b-c)^m$
$6n-1$	$\Pi(b-c)\,.\,(a^2+b^2+c^2-bc-ca-ab)$
$6n$	None
$6n+1$	$\Pi(b-c)\,.\,(a^2+b^2+c^2-bc-ca-ab)^2$
$6n+2$	$(a^2+b^2+c^2-bc-ca-ab)$
$6n+3$	$\Pi(b-c)$
$6n+4$	$(a^2+b^2+c^2-bc-ca-ab)^2$.

If $6n\mp 1$ is prime, when there is the further factor $6n\mp 1$, we have the two interesting cases.

4. What about the remaining factor? Method as in Smith, p. 383. Details at the end of these Notes, § 7. The results are :

$$f_{n-1}\equiv\frac{\Sigma(b-c)^{6n-1}}{(6n-1)pq}=\sum_{0}^{n-1} a_r(p^3)^r(q^2)^{n-1-r}, \quad\text{............(4.1)}$$

where $\qquad a_r=(2n+r-1)!/(2n-2r-1)!\,(3r+1)!,$

$$f_{n-1}'\equiv\frac{\Sigma(b-c)^{6n+1}}{(6n+1)p^2q}=\sum_{0}^{n-1} a_r'(p^3)^r(q^2)^{n-1-r}, \quad\text{...........(4.2)}$$

where $\qquad a_r'=(2n+r)!/(2n-2r-1)!\,(3r+2)!.$

We have $a_1=1=a_{n-1}=a_{n-1}'$ and $a_1'=n.$

Hence $\qquad f_{n-1}'-f_{n-1}=q^2\cdot g_{n-2}$ say.

Also

$$a_r'-a_r=\frac{(2n+r-1)!\{(2n+r)-(3r+2)\}}{(2n-2r-1)!\,(3r+2)!}$$

$$=\frac{1}{2n-2r-1}\binom{2n+r-1}{3r+2}. \quad\text{......................(4.3)}$$

Hence $\qquad a_1'-a_1=n-1=a_{n-2}'-a_{n-2}.$

Hence $\qquad (n-1)f_{n-2}-g_{n-2}=p^3q^2\,.\,h_{n-4}$ say.

The general coefficient in this is

$$\frac{(n-1)(2n+r-3)!}{(2n-2r-3)!\,(3r+1)!}-\frac{(2n+r-1)!}{(2n-2r-1)(2n-2r-3)!\,(3r+2)!}$$

$$=\frac{(6n-5)r(n-r-2)}{2n-2r-1}\cdot\frac{(2n+r-3)!}{(2n-2r-3)!\,(3r+2)!}, \quad\text{........(4.4)}$$

on simplification.

Hence h_{n-4} is divisible by $6n-5$ and r ranges from 1 to $n-3$ inclusive.

We also have

$$3a_r' - a_r = \frac{(2n+r-1)!\,2(3n-1)}{(2n-2r-1)!\,(3r+2)!}$$

$$= \binom{2n+r-1}{3r} \frac{2(3n-1)}{(3r+1)(3r+2)} \quad \text{..................(4.5)}$$

and

$$na_r - a_r' = \frac{(2n+r-1)!\,r(3n-1)}{(2n-2r-1)!\,(3r+2)!}$$

$$= \binom{2n+r-2}{3r-1} \cdot \frac{(2n+r-1)(3n-1)}{3(3r+1)(3r+2)} \cdot \quad \text{...........(4.6)}$$

5. *Some properties of integers :*

(*a*) If $6n-1$ is prime, a_r is an integer. Hence $\binom{2n+r-1}{3r}$ is divisible by $3r+1$ and $\binom{2n+r-1}{3r+1}$ by $2n-2r-1$.

(*b*) If $6n+1$ is prime, a_r' is an integer. Hence $\binom{2n+r}{3r+1}$ is divisible by $3r+2$ and $\binom{2n+r}{3r+2}$ by $2n-2r-1$.

(*c*) From (4.3), when $6n-1$ and $6n+1$ are both prime, $\binom{2n+r-1}{3r+2}$ is divisible by $2n-2r-1$.

(*d*) From (4.5), when $3n-1$, $6n-1$, $6n+1$ are all prime, $\binom{2n+r-1}{3r}$ is divisible by $\frac{1}{2}(3r+1)(3r+2)$; and from (4.6),

$$(2n+r-1)\binom{2n+r-2}{3r-1} \text{ is divisible by } 3(3r+1)(3r+2).$$

Some values of n for which all the foregoing results are true are given in the following table :

n	$3n-1$	$6n-1$	$6n+1$
10	29	59	61
18	53	107	109
30	89	179	181
38	113	227	229

6. When $6n-7$ is prime, all the coefficients in f_{n-2} are integers. Hence if, further, $6n-5$, $6n-1$, $6n+1$ are prime, the coefficients in $h_{n-4}/(6n-5)$ are integers, from (4.4).

Some values of n giving four consecutive primes are 2, 3, 18, 33, 138, 313 ; and for these values of n and for $1 \leqslant r \leqslant n-3$

$$\tfrac{1}{6}(2n+r-3)(2n+r-4)(2n+r-5) \cdot \binom{2n+r-6}{3r-1}$$

is divisible by $(2n-2r-1)(2n-2r-3)(3r+1)(3r+2)$.

7. *Detail* for § 4.

Let $\{1+(b-c)x\}\{1+(c-a)x\}\{1+(a-b)x\} = 1 - px^2 + qx^3$,

where $\quad p \equiv -\Sigma(c-a)(a-b) = a^2+b^2+c^2-bc-ca-ab,$

$$q \equiv (b-c)(c-a)(a-b).$$

Then $\quad \Sigma \log\{1+(b-c)x\} = \log\{1-x^2(p-qx)\},$

so that $\sum_2^\infty \{(-1)^{r-1}x^r r^{-1}\Sigma(b-c)^r\}$ is equal to

$$\ldots - \frac{x^{4n}}{2n}(p-qx)^{2n} - \frac{x^{4n+2}}{2n+1}(p-qx)^{2n+1}\ldots$$

$$\ldots - \frac{x^{4n+2r}}{2n+r}(p-qx)^{2n+r} - \frac{x^{4n+2r+2}}{2n+r+1}(p-qx)^{2n+r+1}\ldots$$

$$\ldots - \frac{x^{6n-2}}{3n-1}(p-qx)^{3n-1} - \frac{x^{6n}}{3n}(p-qx)^{3n}\ldots .$$

The coefficient of x^{6n-1} in the right-hand side is

$$-\sum_0^{n-1} \frac{1}{2n+r}\binom{2n+r}{2n-2r-1} p^{3r+1}(-q)^{2n-2r-1}$$

and the coefficient of x^{6n+1} is

$$-\sum_0^{n-1} \frac{1}{2n+r+1}\binom{2n+r+1}{2n-2r-1} p^{3r+2}(-q)^{2n-2r-1}.$$

Hence

$$\frac{\Sigma(b-c)^{6n-1}}{(6n-1)pq} = \sum_0^{n-1} \frac{(2n+r-1)!}{(2n-2r-1)!(3r+1)!}(p^3)^r(q^2)^{(n-1)-r}$$

and

$$\frac{\Sigma(b-c)^{6n+1}}{(6n+1)p^2q} = \sum_0^{n-1} \frac{(2n+r)!}{(2n-2r-1)!(3r+2)!}(p^3)^r(q^2)^{(n-1)-r}.$$

N. M. Gibbins.

1395. *A query.*

Can any reader give any information with regard to the problem of finding three square integers in arithmetic progression? One solution is expressed by the formula

$$(m^2+n^2)^2,\ (m^2+n^2)^2 \pm 4mn(m^2-n^2),$$

where m, n are unequal positive integers. It seems probable that this problem has been attempted before.

F. G. Maunsell.

1396. *A query.*

Can any reader give a proof that the equations, $c^6+15=16d^3$, $125c^6+3=16d^3$, $8c^6+1=9d^3$, where c and d are positive integers, have each but one solution, namely (1, 1), (1, 2), (1, 1)?

J. M. Child.

1397. *Perfect numbers.*

(1) 6.

(2) 28.

(3) 496.

(4) 8128.

(5) 33550336.

(6) 8589869056.

(7) 137438691328.

(8) 2305843008139952128.

(9) 2658455991569831744654692615953842176.

(10) 191561942608236107294793378082303638130997321548169216

(11) 13164036458569648337239753604587229102234723183869431177783728128.

(12) 14474011154664524427946373126085988481573677491474835889066354349131199152128.

J. Travers.

EXAMINATIONS SUB-COMMITTEE

An Invitation to Members

This committee has made arrangements for an investigation of the School Certificate papers in Elementary Mathematics set during the summer by the major examining Boards, namely:

University of London,
Oxford and Cambridge Joint Board,
Northern Universities Joint Board,
Central Welsh Board,
Cambridge Local Examinations Syndicate,
Oxford Local Examinations Syndicate.

For each Board the papers will be worked and reviewed by two persons; one of these will already be familiar with the syllabus and its interpretation while the other will not. All comments received, appreciative or otherwise, will then be considered by the sub-committee.

This announcement is intended also to serve as a general invitation to members of the Association to join in this investigation by sending to the secretary of the sub-committee any comments upon these papers which they may wish to make. These should be received not later than September 11, 1939.

The sub-committee is also considering more general suggestions that might form the basis of a policy for constructive modification of the mathematical examinations at the School Certificate.

C. T. Daltry (Secretary).

99 Maze Hill, London, S.E. 10.

REVIEWS.

Diophantine Equations.

Diophantische Gleichungen. By T. Skolem. Pp. ii, 130. Rm. 15. 1938. Ergebnisse der Mathematik, Band V, Heft 4. (Springer, Berlin)

This recent book has suggested to me that an essay review may be of interest to the general reader.

A diophantine equation may be defined as follows. Let $f(x_1, x_2, \ldots, x_m)$, or say f for shortness, be a polynomial in m variables $x_1, x_2, \ldots, x_m$ with integral coefficients. Required the integral values of the x's which satisfy $f=0$, or say the integral points on the variety or manifold $f=0$. The problem of finding the rational points—that is, $x_1, x_2, \ldots, x_m$ all rational numbers—is reduced to the preceding one but with $m+1$ variables by putting $x_1 = y_1/y_{m+1}$, etc. There is the obvious extension when the single equation $f=0$ is replaced by a system of equations $f_r=0$ $(r=1, 2, \ldots, n<m)$. For simplicity we suppose that all the coefficients of all the equations throughout are integers, and that integral solutions are required unless rational solutions are explicitly mentioned.

The subject, then, is co-extensive with almost all of the theory of numbers, and is so vast that many writers have limited their scope by restricting themselves to equations of higher degree—for example, by dealing with only one equation and that of degree higher than two. There is the more justification for this as the results are interesting, important and extensive enough to fill a large volume. Professor Skolem has preferred, however, to include also in his tract equations of the first and second degrees, and so is continually in touch with diverse and important sections of number theory. He gives, in his six chapters, a more or less detailed outline of some of the classical results, the present state of and recent progress in wide sections of the theory. Thus in the first two chapters he treats of linear equations and equations that are linear in all the unknowns. The third is concerned with the representation of numbers by quadratic forms and the fourth with multiplicative equations. The fifth chapter deals with rational points on algebraic curves and the sixth with integral points on algebraic manifolds.

It cannot be expected that with the very limited space at his disposal the author has been able to exhaust the subject. He must also be allowed the liberty of making his own selection. I propose, therefore, to give not only an account of the chief results in his tract, but also to mention some other related parts of the subject.

The first chapter treats of the system of indeterminate linear equations of the first degree :

$$\sum_{s=1}^{n} a_{rs}x_s = b_r \quad (r=1, 2, \ldots, m<n).$$

When the b's are all zero—that is, when the equations are homogeneous—the theory is due chiefly to H. J. S. Smith. In the general case, the results involve the theory of elementary divisors and are associated with the names of Smith and Frobenius. There are also results as to the number of solutions of some equations of this kind when the variables are assumed to be positive, due to Euler, Sylvester and Cayley. Recently the theory of diophantine approximation has made it important to find simple estimates for the upper bounds of the variables in one solution in the homogeneous case, and such are given in the chapter.

Two interesting types of equations studied in the second chapter may be mentioned. The first is the general bilinear equation

$$\sum_{r=1,\, s=1}^{m,\, n} a_{rs}x_r y_s = b.$$

By the theory of linear divisors, this may be reduced to the form

$$\sum_{\lambda=1}^{\rho} e_\lambda x_\lambda y_\lambda = b,$$

where ρ is the rank of the matrix of the a's. The solution takes a very simple parametric form when $b=0$. In particular, the general solution of

$$x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$$

is given by

$$x_1 = u_1 v_2 - u_2 v_1, \quad x_2 = u_3 v_1 - u_1 v_3, \quad x_3 = u_1 v_4 - u_4 v_1,$$

$$y_1 = u_3 v_4 - u_4 v_3, \quad y_2 = u_2 v_4 - u_4 v_2, \quad y_3 = u_2 v_3 - u_3 v_2$$

for integral values of the parameters.

The third chapter commences with the theory of the equation

$$f(x_1, x_2, \ldots, x_n) = 0,$$

where f is an *indefinite* homogeneous quadratic form in the n variables. Without loss of generality, the equation may be taken in the simpler form

$$\sum_{r=1}^{n} a_r x_r^2 = 0.$$

When $n=3$, necessary and sufficient conditions for solvability were given by Legendre and also proved by Gauss and Dirichlet, who found all the solutions; and when $n=4$, by Meyer, who also showed that the equation was always solvable when $n>4$. These results were put in a beautiful form by Hasse, who showed that the equation $f=0$ was solvable with all the x's not zero if, and only if, the congruence

$$f(x_1, x_2, \ldots, x_n) \equiv 0 \pmod{p^\lambda}$$

is solvable with all the x's not divisible by p^λ for all primes p and exponents $\lambda > 0$. This implies only a finite number of conditions since the congruence is solvable when p is odd and does not divide the discriminant of f.

Hasse's results have proved fundamental in the recent theory of quadratic forms developed by Siegel. Lack of space prevents Skolem from giving any account of this. He gives, however, the usual account of the representation of numbers by binary and ternary quadratic forms. He also finds room for an account of Hermite's method of reducing indefinite forms and the simplification due to the present writer of the proof by Stouff and Selling of the finiteness of the class number.

Among the general equations included in the fourth chapter is

$$f(x_1, x_2, \ldots, x_n) = h,$$

where f splits into n linear factors. In particular, when f is the norm of a linear form

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n,$$

where the α's are algebraic numbers in a field of degree l, the theory of ideals in an algebraic field is adequate for a complete solution if $n=l$. When there are less than l variables, the question becomes a very difficult one and is referred to later.

Then there are the equations of the type

$$f(x_1, x_2) = y^m,$$

where f is a homogeneous polynomial in x_1, x_2 of degree r. Simple cases such as

$$ax^2 + by^2 = z^3$$

had been dealt with by Euler, *e.g.* in his *Algebra*. On putting

$$x\surd a+y\surd(-b)=\{p\surd a+q\surd(-b)\}^3$$

he found the parametric solution

$$x=ap^3-3bpq^2,\quad y=3ap^2q-bq^3,\quad z=ap^2+bq^2.$$

It is, of course, neither obvious nor true in general that all the integral solutions are obtained from integral values of p and q. The equation really involves the study of algebraic numbers and so for the most part was beyond Euler's power. It is surprising, however, how Euler flounders in his work. Thus he considers that he has proved that the only integer solutions of

$$y^2+2=z^3$$

are $z=3$, $y=\pm 5$, obtained by putting $a=2$, $b=1$, $x=1$ in the parametric solution. This gives $1=2p^3-3pq^2$, whence, as p and q are supposed to be integers, $p=-1$, $q=\pm 1$. By a similar argument with $a=5$, $b=7$, $y=1$, he shows that $5x^2+7$ should never be a cube. He sees, however, that $p=\frac{1}{2}$, $q=\frac{1}{2}$ suffice and give $x=2$ and then states, without proof (which he probably never had), that there are no other values of p, q. It is a little surprising that only many years after the development of the theory of algebraic numbers was it applied to such equations.

When $m=2$, $r=3$ the problem of finding the rational solutions is trivial. The complete integral parametric solution was given by the present writer, who expressed x, y by a finite number of homogeneous quartics in two integral parameters p, q, and z by homogeneous sextics.

In chapter five it is to be expected that the question of the rational points on a curve $f(x, y)=0$, or say the integral points on the curve $f(x, y, z)=0$ in homogeneous coordinates, will involve geometric, algebraic and arithmetic theory, and that it may be convenient to use terminology and illustrations from these sources. Thus two curves $f_1(x, y)=0$, $f_2(\xi, \eta)=0$ are called equivalent if the coordinates x, y are rationally expressible with rational coefficients in terms of ξ, η and conversely. Then obviously the rational points on one curve are known when those on the other are found.

When the curve is of genus zero, the problem of finding the rational points was completely solved by Hilbert and Hurwitz in a joint paper and by Poincaré, although they were all really anticipated by Noether. According as the curve is of odd or even degree, it is equivalent to a straight line or a conic. The rational points on a line are obvious. A conic does not have rational points unless conditions equivalent to those due to Legendre are satisfied, and then the points are easily found. But a conic always has an infinity of rational sets of points of order two—that is, pairs of points whose symmetric functions are rational numbers.

Poincaré showed that any curve of genus one is equivalent to a cubic without a double point. If a cubic has a rational point, it is equivalent to a cubic whose equation has the standard Weierstrassian form

$$y^2=4x^3-g_2x-g_3.$$

The simple proof of this published by Nagell in 1929 is given by Skolem, but the first proof seems to have been given by the present writer in 1912.

The rational points on cubic curves have been for some centuries a favourite subject of investigation for mathematicians of all kinds, both experts and amateurs. When any rational points, say P, Q, ..., are known, for example by inspection, other rational points can, in general, be found by a process which

has been known for at least three hundred years. Thus, expressed geometrically, the tangent at P, unless P is a point of inflexion, meets the curve in another point P_1 whose coordinates are rational since they are determined by an equation of the first degree. Similarly the chord PQ will meet the cubic in a rational point R, in general distinct from P and Q. By combining these two processes, an infinite number of rational points will in general be found, though it is possible that the process may lead to only a finite number.

The geometrical process has a simple analytical interpretation. On using an elliptic parameter, say

$$x = \wp(u), \quad y = \wp'(u),$$

it amounts to saying that if r rational points of parameters $u_1, u_2, \ldots, u_r$ are given, then the points with parameters $m_1 u_1 + m_2 u_2 + \ldots + m_r u_r$, where the m's are any integers, positive, negative or zero, are also rational. This is clear from the addition formula for the elliptic functions.

Poincaré conjectured that all the rational points could be found in this way from a finite number of fundamental solutions, and so if R is the minimum number, the solutions define a module of finite basis with rank R. This was proved by the present writer in 1922. An upper estimate for the value of R was given by Billing in 1938, but it is not easy to find curves with an assigned R or, say, rank R.

Curves with $R = 0$ have been known for a long time—for example, the equation (x, y, z not all zero)

$$x^3 + (9a+2)y^3 = 9z^3,$$

where a is any integer, has no solutions, as is easily seen on taking residues mod 9. Lagrange proved that $R = 1$ for the equation $2x^4 - 1 = y^2$ and so this holds for the equivalent equation $y^2 = 4x^3 + 2x$. Recently Faddejév has found interesting results on the rank of the curve $x^3 + y^3 = Az^3$, where A is a prime. Thus when $A \equiv 5$, 11 (mod 18), $R = 1$, $u_1 = 0$ and the only solution is $x = -y$. This was proved first by Lucas and Sylvester. When $A \equiv 7$, 13, 17 (mod 18), $R = 1$, and there will be an infinity of solutions if u_1 is not a submultiple of a period; and when $A \equiv 1$ (mod 18), $R \leqslant 2$. Billing has also given numerical results.

An allied question is that of finding cubics with an assigned number N of rational points. Curves with $N = 0$, 1 have been given above and curves are known with $N = 2$, 3, 4, 6, ∞ but the general problem has not been solved.

It may be remarked that recently Fueter and his pupil Brunner have broken new ground in giving general equations of the type $x^3 - D = y^2$ (associated with the names of Bachet, Fermat and Euler), which have no integral solutions—for example, in the case when D satisfies the conditions :—

$$D > 0,\ D \equiv 7 \pmod 9,\ D \not\equiv -1 \pmod 4,\ D \not\equiv -4 \pmod{16},$$

D does not contain an odd prime to an even power and the class number of the quadratic field $k(\sqrt{-D})$ is not divisible by 3. Previous results, for example those by Nagell, assumed that D had special forms.

Very little is known about the rational points on curves of genus $p > 1$. It has been conjectured that there are only a finite number of rational points on any such curve. For rational sets of points of order p on the curves—that is, sets of p points whose symmetrical functions are rational—Weil generalised Mordell's result for $p = 1$.

The best known among equations with $p > 1$ is that associated with Fermat's last theorem, which asserts that if $n > 3$, the equation

$$x^n + y^n = z^n$$

has no integer solutions with $xyz \neq 0$. Undoubtedly this equation has been most important in the history of mathematics in that it led Kummer to the discovery of ideal numbers from which arose the theory of algebraic numbers. Skolem, however, has very little to say on this equation, preferring to refer to Bachmann's book. He is, moreover, not quite up to date. It is sufficient to suppose that n is a prime number p and then two cases arise according as $xyz \not\equiv 0$ or $\equiv 0 \pmod p$. For solutions to exist in the first case, Skolem mentions Furtwängler's necessary condition that every prime divisor r of xyz (and also of $x \pm y$, etc., not mentioned by Skolem) satisfies the congruence $r^{p-1} \equiv 1 \pmod{p^2}$, and since r may be taken to be 2 or 3, earlier results of Wieferich and Mirimanoff follow. Skolem is not aware that this congruence holds for other values of r than divisors of x, etc.—for example: 5, proved in 1914 by Vandiver; 11, 17 and when $n \equiv 2 \pmod 3$ 7, 13, 19 proved by Frobenius in 1914. It is an unsolved question to find all the values of r for which the congruence is a necessary condition in the first case.

The last chapter, on the integral points of curves, commences with an account of the work of Runge on the integral solutions of $f(x, y) = 0$, where f is a polynomial of degree n, irreducible over the field of the rational numbers. Runge gives the first general necessary (but not sufficient) conditions for the existence of an infinity of integral solutions. He deduces, however, some important corollaries. Thus if the homogeneous part of degree n of f is not a power $\geqslant 1$ of an irreducible polynomial, there is only a finite number of integral solutions. His method depends upon expanding y in a series of descending powers of x, but a simpler presentation is due to Skolem. Applications have been made by Thue and Skolem to results such as: if $f(0, 0) = 0$ and the greatest common factor of x and y is bounded, then there is only a finite number of solutions. Nagell has also applied the method to the equation $y^2 = (x^n - 1)/(x - 1)$ and has shown that if $n \geqslant 3$, then $|x| \leqslant 2^{n-3}$.

Among the most important results in this last chapter are the theorems associated with the names of Thue (1909) and Siegel. They can be stated as results on diophantine approximation—that is, as estimates of approximations by rational fractions to a real algebraic number ρ defined as a root of an equation $F(x) = 0$, where $F(x)$ is a polynomial of degree r irreducible over the field of the rational numbers. Thus if $c > 0$, $\epsilon > 0$ are given numbers, there are only a finite number of integers p, q such that

$$\left|\rho - \frac{p}{q}\right| < c/q^{\frac{1}{2}r+1+\epsilon}.$$

An immediate deduction on writing $f(x, y) = y^r F(x/y)$ is Thue's important theorem of 1909:

If $f(x, y)$ is a homogeneous irreducible polynomial of degree greater than 2 with integral coefficients and c is a given number, the equation $f(x, y) = c$ has only a finite number of integral solutions.

Siegel showed that the exponent $\frac{1}{2}r + \epsilon + 1$ could be replaced by $2\sqrt{r}$. The ideas involved and the results deduced are of great importance in questions on transcendental numbers. A p-adic extension of these results has been given by Mahler. From the theorems of Thue and Siegel and their generalisations flow many particular results of interest. Thus Thue showed in 1916 that the equation

$$ay^2 + by + c = dx^n, \quad n \geqslant 3, \quad ad(b^2 - 4ac) \neq 0$$

has only a finite number of integral solutions. The present writer proved that

$$ax^3 + bx^2 + cx + d = ey^2$$

had only a finite number of integral solutions. This was an immediate application of Thue's theorem to the present writer's result of 1912 and could have been proved in one line if he had been aware of Thue's result. As it is, Thue has priority of publication for $n = 3$.

These are all particular cases of Siegel's general theorem on the integral solutions of the equation $f(x, y) = 0$, which asserts that there are only a finite number if $p \geqslant 1$. It is not an easy matter to find them or to estimate their number. Some progress, however, has been made of late with special equations, not without a certain generality. Thus a result of Delaunay's states that the equation $x^3 + dy^3 = 1$, where d is an integer, not a cube, has at most one integer solution other than $x = 1$, $y = 0$, and that this is easily found from the usual theory of units in an algebraic field. More generally the work of Delaunay and Nagell shows that the equation

$$ax^3 + bx^2y + cxy^2 + dy^3 = 1,$$

where the cubic has only one real linear factor, has at most three integer solutions, except that there are exactly four solutions when the form

$$(a, b, c, d) \sim (1, 0, 1, 1) \text{ or } (1, -1, 1, 1)$$

and five solutions when $(a, b, c, d) \sim (1, 0, -1, 1)$. Extensions of these results have been made to various equations of the type $ax^n + by^n = c$ by Tartakowski, Ljunggren, Faddejév, and Siegel.

There is an interesting application of results of this kind to a problem not mentioned by Skolem. Let $f(x)$ be a polynomial with integral coefficients and at least two different zeros. Denote by $P(x)$ the greatest prime dividing $f(x)$, where x is an integer and $f(x) \neq 0$. Siegel has proved easily that $\lim\limits_{x \to \infty} P(x) = \infty$ when $f(x) = 0$ has no multiple roots. Mahler has proved that when

$$f(x) = Dx^2 - A,$$

where A is one of ± 1, ± 2, and D is square-free and prime to A, then if x is prime to A, $\overline{\lim} \{P(x)/\log \log x\} \geqslant 1$. Part of this result was found later by Chowla. Nagell showed that a similar result holds for the cubic polynomial $Dx^3 - A$, where A is one of ± 1, ± 3.

The chapter closes with an account of a p-adic method which Skolem himself has recently developed and applied. Its foundations are very old and go back to Euler's proof that the equation $x^3 + py^3 + p^2z^3 = 0$, where p is a given prime, has no integral solutions except $x = y = z = 0$. This is almost obvious since it is easily seen that x, y, z are all divisible by p, whereas they may be assumed to have no common factor. The theory of p-adic numbers is now a part of modern algebra and it has led to results of considerable importance. Thus Skolem uses it to prove Thue's fundamental theorem except in the case when the linear factors of $f(x, y)$ are all real. He applies it also to equations with more than two variables. Let α, β, γ be given linearly independent numbers in a real quintic field whose conjugate fields are all imaginary. He proves that the equation

$$N(\alpha x + \beta y + \gamma z) = h,$$

where N denotes the norm—that is, the product of five conjugate factors—has only a finite number of solutions in integers x, y, z. The proof has been simplified and the results generalised by Chabauty in a Paris dissertation published since the appearance of Skolem's tract.

Skolem's two final chapters indicate that during the last few years substantial progress has been made with non-homogeneous indeterminate equations in two variables and homogeneous equations in three variables. It is very different, however, with equations in more variables, and there are a number of problems awaiting solution not even hinted at in the tract. Thus let $f(x_1, x_2, \ldots, x_n)$ be an indefinite homogeneous form of degree r in n variables. Is it true that when $n \geqslant r^2+1$, the equation $f=0$ always has integral solutions in which all the x's are not zero? Meyer proved it to be so when $r=2$. It is easy to construct equations with $n=r^2$ which have no solutions except $0, 0, \ldots, 0$, as shown by Hasse and the present writer. Then questions such as whether every integer can be expressed as the sum of four cubes, positive, negative or zero, and obvious extensions to other powers, still await solution. Then there is Landau's theorem that every large positive integer can be expressed as the sum of at most eight positive integral cubes, and the question whether the eight can be replaced by a smaller number. Questions of this kind soon become closely related to the problems of *partitio numerorum* and the methods developed by Hardy and Littlewood, and Vinogradoff. Just recently, subsequently to the publication of Skolem's tract, important progress has been made by Davenport, who shows, for example, that almost all integers are the sum of four positive cubes. An important step is that of finding lower bounds for the number of integers less than n expressible as the sum of s positive kth powers, and interesting methods and results have been developed by Davenport and Erdös. The tract makes no mention of the old results but, of course, it would hardly be feasible to give an account of these methods.

With three methods of attack available, namely: analytical methods; arithmetical methods based on modern algebra which has already thrown new light on old questions; and the algebraic and arithmetic theory of functions and function fields of many variables, more progress may be expected in the future. The present account may give some indication of the suggestive treatment and wealth of material considered by Skolem. The reader interested in this subject will find the tract invaluable and indispensable both for reference and for pleasant reading. L. J. MORDELL.

Probability, Statistics and Truth. By R. VON MISES. Translated by J. NEYMAN, D. SHOLL and E. RABINOWITSCH. Pp. xvi, 323. 12s. 6d. 1939. (Hodge)

This is a translation, with some additional matter, of the second German edition. The first edition appeared in 1928. As this was not reviewed in the *Gazette*, it may be worth while to set forth the gist of Professor von Mises' arguments. If they can be sustained, they are of fundamental importance.

The term *probability* is used, and legitimately used, in many ways in ordinary language. We may speak of the probability that Germany may once again be at war with Liberia, or that the biblical narrative is historically accurate, or that an obscure passage of a Latin author has been translated correctly. In none of these cases, according to von Mises, is the mathematical theory of probability applicable, any more than the mathematical definition of work, as the product of force and distance, is applicable to the work performed by an actor in a stage play. The legitimate fields of application of the mathematical theory of probability include games of chance, statistics of sex distribution of births, insurance, errors of observation, telephone traffic, the kinetic theory of gases, quantum theory, and statistical physics generally. The most obvious characteristic of these is that they are mass phenomena, which recur indefinitely. More precisely, mathematical probability applies only to a *collective*, which is defined as an unlimited sequence of observations fulfilling two

conditions: (i) the relative frequencies of particular attributes of single elements of the collective tend to fixed limits; (ii) these fixed limits are not affected by any place selection. This second condition, to which we will recur later, is known as the Principle of Randomness, and is the crucial point of von Mises' theory. After a collective has been defined, and when, as the result of a long series of experiments or observations, we can decide that a certain mass phenomenon really does satisfy the conditions for a collective, then, and not before, we may speak of the fixed limit as *the probability of the attribute considered within the given collective.* Whenever the word probability is used in a mathematical sense, it is to be considered merely as an abbreviation for the above phrase. In short, probability can be estimated only after acquiring a good deal of knowledge, and not, as many have asserted, in the Principle of Insufficient Reason, from complete ignorance. One curious result of the new definition is that an event with probability zero is, though infinitely rare, not absolutely impossible. Similarly, probability unity is not the same as absolute certainty.

It will be seen that von Mises' theory can be described briefly as the frequency theory of probability presented by Venn as early as 1866 and held by Chrystal and others, but rejected by Keynes, with the essential addition of the Principle of Randomness. This may also be described as the Principle of Impossibility of a Gambling System. It asserts, for example, that in the long run nothing is gained by betting on red only after black has turned up previously. If this principle is accepted, it appears that the whole theory can be built up in a satisfactory manner from the concept of a collective. On the other hand, some assert that random sequences do not exist. Von Mises claims that the investigations of Copeland and Wald have completely disposed of this objection, but it is very doubtful if this claim is generally accepted. Dörge has given a modified form of von Mises' theory which appears to make it impregnable on the theoretical side, at the cost of making it impossible to apply in practice. To sum up, we may say that many authorities regard von Mises' theory favourably, without being fully convinced that it is yet in a wholly satisfactory form. On the other hand, a group of Italian mathematicians, in particular Cantelli, regard it as wholly unsound, on the ground that the limits mentioned in the definition of a collective cannot exist.

The fundamental principles discussed above are contained in the first three of the six lectures of which the book consists. The fourth is more technical, dealing with the Laws of Large Numbers. It is claimed that all of these can be deduced from the author's definitions, whereas, on the classical "equally likely" theory, none can give predictions concerning sequences of observations. Moreover, in the new theory, all the difficulties concerning inverse probability, or the probability of causes, vanish, for this is regarded as a probability in a collective and so on exactly the same footing as the usual probability of events. The fifth lecture deals with applications to statistics and the theory of errors. Death rates are analysed and are considered to indicate a mixture of several collectives. The sixth lecture deals with physical applications, including gas theory, the Brownian movement, entropy, statistical mechanics, and quantum theory. The book concludes with a summary of the whole six lectures, in fifteen propositions, and twelve pages of bibliographical notes. It is intended for non-mathematical readers, and formulae are almost entirely avoided. In the reviewer's opinion, at least half of the book will be intelligible to the general reader with no mathematical knowledge. Those who desire a full development of the subject should consult the author's *Wahrscheinlichkeitsrechnung und ihre Anwendungen* (1931).

H. T. H. Piaggio.

Advanced Mathematics for Engineers. By H. W. Reddick and F. H. Miller. Pp. x, 473. 20s. 1938. (John Wiley, New York; Chapman & Hall)

According to the authors, this book has evolved from courses given by them at the Cooper Union Institute of Technology, New York, and is designed to show some of the various rôles played by advanced mathematics in engineering technology. Topics of value to all branches of engineering have been included and emphasis has been placed throughout on physical applications by presenting, with each principal topic, problems relating to the four main fields of engineering.

The book begins with a chapter on ordinary differential equations, dealing with the usual types of first order equations, equations of higher order with one variable absent, and linear equations with constant coefficients. The applications cover a wide field, including problems in dynamics, the flow of heat, chemical solutions, electric circuits, bending of beams, etc. Then follows a chapter on hyperbolic functions, with applications, including the theory of Schiele's pivot, a uniform transmission cable with direct current, rotating shafts, etc.; and in Chapter III we come to elliptic integrals. These are introduced by way of the pendulum and the rectification of the ellipse. The integral of the third kind and the Jacobian functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ are mentioned, but the theory is not developed and the remainder of the chapter is devoted to discussion of the problems of the capillary rise between two vertical plates, the elastica, the swinging cord, and the field intensity due to a circular current.

In a book intended for engineering students, who are mainly interested in how the mathematical theory can be utilised in their work, too much insistence on rigorous proofs of the conditions of validity of infinite processes is out of place, yet the students must be made aware of the necessity for some care in their use. The authors have therefore compromised in Chapter IV by giving a brief discussion of convergence, including the ratio test and Weierstrass' M-test (which are sufficient for most practical purposes), and recapitulating, in the form of theorems given without proofs but illustrated by examples, the conditions under which the usual operations with infinite series are valid. Incidentally, in their attempt to be as brief as possible, they have fallen into error on p. 146, where they imply that the existence of derivatives of all orders at $x=x_0$ is a sufficient condition for a function to be expanded in the form of a Taylor's series. The chapter then gives applications of infinite series in the calculation of definite integrals and brief sketches of Picard's method of successive approximations and Frobenius' method of solution in series.

Chapter V, on Fourier series, follows the usual lines. Reference is made to the use of Fourier series in suspension bridge theory, but most of the applications are deferred to later chapters. The Gamma and Bessel functions are introduced in Chapter VI. The series for $J_n(x)$ is obtained from the differential equation and the formulae connecting J_0 and J_1 and the ber and bei functions are obtained by manipulating the series. The Bessel functions of the second kind are not discussed, but the Fourier-Bessel expansion of a function in terms of $J_0(\alpha x)$ is given. Again the applications mainly appear later. Partial differentiation is dealt with in Chapter VII. The principal equations of mathematical physics are dealt with and the method of separating the variables and building up solutions to fit boundary conditions illustrated by examples, amongst which some problems of the flow of heat, the flow of electricity in a cable and eddy currents in the core of a solenoid are discussed in detail.

Chapter VIII deals briefly with the elements of vector algebra and calculus, developing sufficient of the theory to lead up to the theorems of Gauss, Green

and Stokes and their applications to potential theory and the flow of heat. Maxwell's electromagnetic field equations are also briefly discussed. Chapter IX, on probability, deals with the combination of probabilities, discusses the normal curve of error, defining probable error and standard deviation, and concludes with a section on the method of least squares.

The next chapter, on functions of a complex variable, is divided into three parts. In Part I, analytic functions are defined and the Cauchy-Riemann equations obtained, mapping by functions of a complex variable is dealt with and the question of multiple valued functions touched on. Part II is devoted to the Schwarz-Christoffel transformation and its applications ; and Part III, which is intended as an introduction to the second part of Chapter XI, deals with line integrals, Green's lemma, Cauchy's theorems and the calculation of integrals by means of the theory of residues.

Chapter XI, on operational calculus, is very well done. The subject is introduced by way of the operator Q, which, acting on a function of t, gives the definite integral from 0 to t ; and after some equations have been solved with its aid, the Heaviside operator p is introduced. This definition of p as the inverse of Q gives point to the necessity for care in interpreting p as d/dt. Heaviside's methods of dealing with electrical networks are then carefully described and illustrated by examples. In the second part, some problems involving partial differential equations are solved with the aid of Bromwich's line integrals.

As is inevitable in a book touching on so many topics, there are points where the treatment is rather sketchy, but for the main part the theory is developed with great care and well illustrated by the applications, so that the book is a useful introduction to some important branches of advanced mathematics. Problems to be worked by the student are provided throughout the book, but it must be remarked that in several instances these appear to be merely numerical calculations based on results obtained in the text. The physical principles involved in the applications are clearly stated and no great familiarity with technical terms is required to follow them. Mathematicians engaged in giving advanced courses in their subject to engineering students should find this a very useful book. W. H.

Higher Mathematics (with Applications to Science and Engineering). By R. S. Burington and C. C. Torrance. Pp. xiii, 844. 30s. 1939. (McGraw-Hill)

The above book is on pure mathematics, more particularly on calculus, algebra and analysis. Physicists or engineers who wish to enlarge their knowedge beyond the bounds usually set by textbooks which are labelled " elementary " will find this book useful. Even those who are interested only in the processes of pure mathematics described in the book will find much of interest in it, though the book does not come up to the standard of mathematical style and rigour set by Whittaker and Watson's *Modern Analysis.* This contrast is not intended to deprecate the volume at present under review, but rather to help readers to form a picture of the book by contrasting it with another which is well known to all who teach mathematics or are engaged in mathematical research. Even this comparison is somewhat unfair, for the preface makes it quite clear that the authors did not intend to write a book comparable with the classical " Whittaker and Watson ". In scope, however, there is a great deal of similarity. In comparing the two books it must be borne in mind that Burington and Torrance assume a much lower initial level of mathematical knowledge than do Whittaker and Watson. *Higher Mathematics* covers much the same ground as *Modern Analysis* and a great deal more, but the limitations imposed by the mathe-

matical knowledge of the students for whom the book is intended, and the already ample dimensions of the volume, make it impossible for the authors to enter into detailed mathematical discussion in the manner of the writers of *Modern Analysis*.

The preface states that " This book is an outgrowth of a series of courses in advanced calculus and related subjects, given by the authors at the Case School of Applied Science, and designed primarily to meet the growing needs of students interested in the applications of mathematics to physics and engineering. To this end, special care has been taken to emphasize physical meanings of notations and relationships occurring in the subject. Applications to many branches of physics and engineering are given. These applications have been included as integral parts of the explanations of the several mathematical topics, and exercises involving them will be found in every chapter." It is also stated in the preface that the book is suitable for students of pure mathematics. Students in honours schools of mathematics in English universities would not find the book a suitable one to take them through their studies. There are other books, much more suited to the needs of pure mathematicians, which the reviewer would recommend in preference to the book by Burington and Torrance.

This book is divided into nine chapters. The first four are devoted to the theory of real variables. Chapter I is a review of the elements of differential calculus. Starting with the concept of the function it works its way through continuity, derivatives, Rolle's theorem, L'Hôpital's rule and Taylor's theorem. The second section of the first chapter deals with partial differentiation and generalises the work of the first section to functions of several variables. The last topic in the chapter is a rather bald and confessedly incomplete account of the maximum and minimum values of a function of two real variables.

Sections A and B of Chapter II deal with elementary integration, line integrals, surface integrals, Green's theorem, Stokes's theorem and the transformation of double integrals. Subsequent sections of the same chapter deal with the Riemann theory of integration and special functions defined by means of definite integrals. There is also a short section of eight pages dealing with numerical integration.

Chapter III deals with elementary ordinary differential equations and, very briefly, with Legendre and Bessel functions. The chapter contains a short section of seven pages dealing with the numerical solution of differential equations. The summation of infinite series, convergence and Fourier series are dealt with in Chapter IV. Chapter V introduces complex numbers and proceeds to a discussion of the integral calculus of complex variables, singularities of analytic functions, conformal mapping and elliptic functions. Chapter VI is concerned with algebra, vector analysis, some differential geometry and some tensor analysis. Chapter VII deals very briefly with partial differential equations. The remaining two chapters, which deal with calculus of variations, analytical dynamics and the elementary theory of real variables, are even more sketchy than the preceding ones.

There are a number of points of general interest which might be made here. The diagrams in the book are neat and clear. The printing is, on the whole, good, though the text contains a number of incredibly ugly symbols of incredible size and blackness. The use of the symbol $p(u)$ for the Weierstrassian elliptic function in the place of the older $\wp(u)$ may be considered, by some readers, to be an improvement in that it makes for greater simplicity and less strain on the eyes. More can be said, however, against certain other experimental changes in symbolism than in their favour. For example, there

is, in the opinion of the reviewer, little or no point in using the symbol ∂ in the expression $\iint f(x, \alpha)\,dx\,\partial\alpha$. The examples in the book are numerous, well chosen and, on the whole, easy. They lay stress on manipulative ability and on the use of intelligence. They are admirably suited to the students for whom the book was written.

In spite of its title a considerable portion of the book is devoted to elementary mathematics. This portion could have been printed separately as an " Introduction to Higher Mathematics ". Such a procedure would have left space in *Higher Mathematics* for the filling up of gaps which are as obvious to the authors as they are to the reader. Sections which might have benefited by fuller treatment have had to be curtailed and have therefore lost in clarity. In spite of this the book is an interesting one and has many important qualities to recommend it. One of its most admirable qualities arises from the fact that the authors are aware that engineers and physicists are growing to appreciate, more and more, the importance of mathematical rigour. L. R.

Grundbegriffe und Hauptsätze der Höheren Mathematik, insbesondere für Ingenieure und Naturforscher. By G. Kowalewski. Pp. 156. Rm. 5. 1938. (Walter de Gruyter, Berlin)

This book is written in Professor Kowalewski's usual lucid and interesting style. As is made clear by the title, and emphasised in the preface, the aim is to present *ideas* rather than methods or applications. This is done in three chapters, on Vectors and Determinants, Laws of Limits, and Differential and Integral Calculus respectively, of which the last is rather more than half the book. There is, however, very little in the way of examples illustrating the ideas and it is doubtful, in view of this, whether the book would appeal to many of those to whom it is addressed. Indeed a similar criticism of the lack of illustrative example, either general or of application, appears in at least one review of the same book from Germany, where the practice of indicating the relevance of mathematics to engineering or science is even less prevalent than it is in this country.

The mathematician, and especially the teacher, may find the book interesting as a concise exposition of fundamental ideas, and perhaps even more by reason of the occasionally unusual order in which the topics are treated, although each one grows naturally out of the preceding. W. G. B.

Theoretical Mechanics. By C. J. Coe. Pp. xiii, 555. 21s. 1938. (The Macmillan Company, New York)

The sub-title of this book is " A Vectorial Treatment ", and this immediately suggests two queries. The first is whether mechanics can be taught best by using vector methods and notation. If the answer to this is in the affirmative, the second query is whether the teaching of vector analysis should be considered part of the work of the applied mathematician, or should he be able to assume that his students are as familiar with this as they are with the differential and the integral calculus. The fundamental vector is essentially a geometrical quantity and vector analysis involves no more than geometry, algebra and calculus. Granted that the most important applications are to physical problems, this is equally true of many branches of pure mathematics.

This book is Professor Coe's answer to the first query ; and because at present the pure mathematicians are in general so absorbed in other branches of the subject that they do not concern themselves much with vectors, the book contains an adequate treatment of so much of the theory as is required for subsequent applications. Following the American practice the notation

used is that of Gibbs—a dot for the scalar product and a cross for the vector product.

The book opens with two preliminary chapters, one devoted to the fundamental postulates of classical mechanics and the other to the motion of a particle in a straight line, a fair knowledge of calculus being assumed. The chief properties of free vectors and their products, and of the vector function of scalar time, are then introduced; and on this function is based the treatment of the kinematics of a particle and of a rigid body. The next chapter, devoted to sliding vectors, leads to chapters on statics. The Principle of Virtual Work then appears.

In the two chapters which follow, dealing with the kinetics of particles and of rigid bodies, use is made of the linear vector function, and the moment of inertia is the scalar product of this function and of a unit vector. The work on dynamics closes with a treatment of the variational methods associated with the names of Laplace, Hamilton and Gauss. The two remaining chapters are devoted to vector calculus and to a short discussion of potential theory, attractions and harmonic functions.

The order in which the different branches of the subject are introduced is the natural one viewed from the point of vector analysis, but it has little relation to the traditional order. Thus on p. 107 there appears a particle moving in a twisted cubic curve, on p. 168 there is the definition of the Euler angles, but on p. 229 the problem of a particle at rest acted on by three forces.

As the title indicates, the bias of the book is towards the theoretical side rather than the practical, and this may perhaps explain why such an important topic as the theory of small oscillations is not included. Many of the examples show the same leaning; and although the author states that a considerable number of exercises are offered, such collections as sixteen on the statics of a rigid body and four on the motion of a top hardly seem adequate. Answers are given—in one question on the catenary to seven figures.

Within the bounds laid down by the author the book serves its purpose admirably, and it is a collection of material not easily available in such a logical and compact form elsewhere. It is printed in the United States on a very lightly tinted paper, and no praise would be too high for the workmanship. The pages are a pleasure to look at and remarkably easy to read.

R. O. S.

A School Algebra. By A. Walker and the late G. P. McNicol. Pp. 272. With answers. 3s. 1938. (Longmans)

According to the legend, any Scot who has ability emigrates to England and speedily finds a seat in high places. Consequently—following the legend to its logical conclusion—these gaps caused by the departure of the cream of the people must either be filled by the mediocre remnant or by foreigners. Certainly the mathematical textbooks used in Scottish schools during the past sixteen years have been nearly all imported from England and written by Englishmen. Hall and Knight, Hall and Stevens, Baker and Bourne, Godfrey and Siddons are as well known to the Scot who was at school thirty years ago as Durell and Robson, Durell and Wright, Siddons and Daltry, Siddons and Hughes are known to the Scot who is at school to-day. However, during the past few months there has been a number of textbooks published by Scottish authors, as though the " mediocre remnant " were determined to capture the home market. This Algebra book, written by two members of the staff of the Training College, Glasgow, is the fourth Scottish book reviewed in these columns in the last three numbers of the *Gazette*.

It is definitely a very good book indeed—and this is praise from one who has

in the past been very scornful of anything appertaining to any Training College. The authors have been content to write a book for the first three years of the secondary course and have not tried to cater for the advanced division or higher elementary classes at the same time. In this they have shown commendable wisdom, as the two types of school need different treatment. The plan that they seem to have followed is that once a pupil has been introduced to the four main operations, then he should not devote his attention to one particular section *ad nauseam*, but that there should be frequent changes in his studies. For example, he should not keep at simple equations until he has run the whole gamut of very simple equations, simple equations, not-so-simple equations, difficult-simple equatons, problems leading to simple equations. Rather let him study the first two parts of this section, then do a little factorising, a little manipulation of formulae, a little simplification of fractions, and then back to not-so-simple equations, etc. All the teaching in the book is by means of examples, and these are drawn from a very wide field. Many of them involve the use of simple geometrical principles; and this is particularly good news in Scotland, for the Scottish Education Department's slogan in the past has been that Geometry is Geometry and Algebra is Algebra and the twain shall only meet when the subject is called Trigonometry. Any book which shows a pupil that there are no water-tight compartments in Mathematics is at least working on sound educational principles.

Before reviewing this book I set the following test questions: 1. How is the subject introduced? 2. How are negative numbers treated? 3. How are quadratic equations explained? 4. Is there a large number of good examples? The authors introduce the subject by considering problems in which numbers are used and then the same problems with letters in place of the numbers; in other words, they treat Algebra as generalised Arithmetic. From the teaching point of view the introduction of negative numbers is one of the most difficult parts of the subject, and particularly to convince a pupil that $(-4)\times(-3)$ is equal to $(+12)$. Messrs. Walker and McNicol start in the usual way by considering a thermometer, and thereafter they proceed to hammer home the points by means of temperatures, bank balances, heights above and below sea level, North and South latitude, East and West longitude and B.C. and A.D. Subtraction is treated as complementary addition; thus, What number must be added to $(+3)$ to give $(+9)$? Answer $(+6)$, *i.e.*

$$(+9)-(+3)=(+6).$$

What number must be added to (-3) to give $(+9)$? Answer (12), *i.e.*

$$(+9)-(-3)=(+12).$$

Therefore, $\quad -(-x)=+(+x)$ and $-(+x)=+(-x)$.

I do not think that the crucial point in the solution of quadratic equations is given sufficient emphasis—that is, if $(x-3)(x-2)=0$, then why is $x=3$ or 2? Also when simultaneous quadratic equations are explained surely the two equations $x^2+xy=10$ and $y^2-xy=3$ are not homogeneous. They are solved by the formation of a homogeneous equation, but they are not homogeneous themselves. Finally, the fourth "test" question must be answered with enthusiasm. There must be well over two thousand examples, including a hundred revision papers, and the questions are varied and well chosen. It is to be regretted that Mr. McNicol has died since the book was written, for there is room for a sequel and this "team" seems to have been well qualified to write it. In conclusion, if this is a sample of the work that the stay-at-home Scot can do, then it is surprising that the market should be almost entirely in the hands of English authors. A. I.

Variationsrechnung im grossen (Morsesche Theorie). By H. SEIFERT and W. THRELFALL. Pp. 115. Geb. Rm. 6.75 ; geh. Rm. 6. 1938. Hamburger mathematische Einzelschriften, 24. (Teubner)

This book is a first-class introduction to the topological part of Marston Morse's book, *The Calculus of Variations in the Large* (reviewed in the *Math. Gazette*, XIX, 1935, pp. 236-7), stopping short of the difficult chapter on closed extremals. In Chapter I, after the necessary preliminaries on topological spaces, singular cycles and the like, a critical value of a function J, in a given topological space Ω, is defined in purely topological terms. A value α is said to be critical if some relative cycle in the set $\{J \leqslant \alpha\}$, mod $\{J < \alpha\}$ in the sense of Lefschetz, is non-bounding mod $\{J < \alpha\}$. The kth connectivity of $\{J \leqslant \alpha\}$ mod $\{J < \alpha\}$ is called the *kth* type number of a critical value α. Subject to a certain condition (Axiom I, p. 24), which is satisfied in many of the most interesting applications, it is shown that $M^k \geqslant R^k$, where M^k is the sum of the kth type numbers of the critical values of J and R^k is the kth connectivity of Ω, with the possibility that both may be infinite. Subject to a second axiom (p. 26) and a certain restrictive condition on the function J it is shown that $M^k = R^k$. It should, perhaps, be said that there are many spaces which are incapable of carrying a function satisfying this last condition.

Chapter II has to do with critical values and critical points of twice-differentiable functions of n real variables. The main result is the fundamental theorem which states that if

$$J = \alpha - x_1^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_n^2 + \chi(x_1, \ldots, x_n),$$

where χ is of the third order near the origin, then the latter is a critical point whose i^{th} type number is unity and all the others zero. If the origin is the only stationary point for which $J = \alpha$, the same applies to the type numbers of the critical value α. These two chapters lead up to Chapter III, which contains an account of a typical problem in the calculus of variations in the large—namely, to determine the number of geodesic segments AB of a given " type ", where A and B are points in a closed Riemannian manifold M^n. If M^n is an n-sphere in Euclidean $(n+1)$-space this problem can be solved by actual calculation, and it follows from the part of the theory developed here that, if M^n is homeomorphic to an n-sphere, then there is at least one geodesic segment AB of each type. The chapter concludes with an account of the similar problem where A and B are replaced by sub-manifolds, the geodesic segment joining them to be a common normal.

There is an appendix on the critical points of a twice-differentiable function J, defined over a closed manifold M^n. Marston Morse's inequalities of the form

$$M^k - M^{k-1} + \ldots \pm M^0 \geqslant R^k - R^{k-1} + \ldots \pm R^0$$

are obtained on the assumption that J has but a finite number of critical points, which may be degenerate. In case the critical points are non-degenerate the above definition of the type numbers leads to a particularly simple proof of these inequalities. The appendix concludes with a proof of the theorem, due to Lusternik and Schnirelmann, that the actual, as apart from the algebraic, number of critical points of J is at least as great as the category of M^n.

To summarise, the book contains a self-contained account of critical values of a function, whose argument may be a curve in a given manifold, their type numbers and their relation to the topology of the space over which the function is defined. This theory is applied in detail to the study of a typical problem in the calculus of variations in the large. The exposition is brilliantly clear and the book is altogether pleasing. J. H. C. W.

La Topologie des Groupes de Lie. By ELIE CARTAN. Pp. 28. 10 fr. 1936. Actualités scientifiques et industrielles, 358 ; exposés de géométrie, VIII. (Hermann, Paris)

This book contains a survey of what is known of the subject, most of which knowledge is due to Prof. Cartan himself. After two chapters on the generalities of the theory there is a chapter on closed groups and one on closed simple groups. It is easy to see that if a simple group is closed its fundamental quadratic form is definite. The converse is true but not so easy to prove. It follows from Weyl's theorem that the universal covering group covers the (closed) adjoint group with a finite degree, and is therefore closed. A proof of this theorem is outlined on pp. 11 and 12. There follows a chapter on open groups, containing the theorem that the first Betti number of an open simple group is 0 or 1. In the next chapter (Chapter VI) there is a very pretty proof of the third fundamental theorem of Lie, asserting the existence of a group with given constants of structure. Of course this was proved by Lie himself for what is now known as a " gruppenkeim " (in general a " gruppenkeim " is not a group, since a product xy will only exist if x and y lie near enough to the identity). But it was left for Cartan himself, in 1930, to prove the existence of an actual group with given constants of structure. The book concludes with a statement of all known theorems on the Betti numbers of closed simple groups—among others the results of L. Pontrjagin and R. Brauer, who have calculated them for the four main types of simple group. J. H. C. W.

Sur les espaces à structure uniforme et sur la topologie générale. By A. WEIL. Pp. 40. 15 fr. 1937. Actualités scientifiques et industrielles, 551. (Hermann, Paris)

The object of this book is to abstract the notion of uniformity, as applied to continuity and convergence, from that of distance. Of course the topology of a metric space is not a metric geometry, like Euclidean or Riemannian geometry, since two metrics are regarded as equivalent if they lead to the same definition of closed sets. But the possibility of defining a given topological structure by means of a metric is a restriction. In the case of a locally compact space, for example, it is equivalent to the second axiom of countability—" malfaisant parasite qui infeste tant de livres et de mémoires dont il affaiblit la portée tout en nuisant à une claire compréhension des phénomènes ".

The characteristic features of a *uniform* topological structure are that an index, ranging over a given set of elements, is associated with each neighbourhood $V_\alpha(p)$, where p is any point and α is the index, and that

A tout indice α on peut faire correspondre un indice β tel que les deux relations $p \epsilon V_\beta(r)$, $q \epsilon V_\beta(r)$ entraînent $q \epsilon V_\alpha(p)$.

Thus, in a metric space, one can take $V_\alpha(p)$ to be the set of all points whose distances from p are less than α, the values of the indices being positive numbers. A topological group provides another example of a uniform structure. Here one starts with a set of neighbourhoods V_α, of the unit element, and if x is an arbitrary element one may define $V_\alpha(x)$ either as the set $V_\alpha x$ or as the set xV_α.

As an alternative statement of the conditions for a uniform structure consider the topological product $E^2 = E \times E$ of a space E with itself, a point in E^2 being an ordered pair of points in E. Let Δ be the " diagonal locus " in E^2, consisting of the points (p, p), and let V_α be a family of sets in E^2 whose intersection is Δ. Then $V_\alpha(p)$ is defined as the set of all points, q in E, such that (p, q) is in V_α. The sets V_α satisfy certain conditions, given on p. 8, including a condition corresponding to the one quoted above. This is equi-

valent to the following: to each α shall correspond a β such that (p, q) is in V_α provided there is a point r, in E, such that (r, p) and (r, q) are in V_β. It is proved later (p. 16) that the space E is topologically equivalent to a metric space if, and only if, these neighbourhoods V_α, of Δ, are equivalent to an enumerable set.

This summarises the first section of Chapter I, under the title " Définitions et premiers exemples ". The book then proceeds to develop these ideas, extending to spaces with this uniform structure many of the fundamental theorems concerning metric spaces. For example, Theorem 1 (p. 13) states that every uniform space E is " completely regular ", meaning that, given a point p and a closed set F in $E-p$, there is a function f, defined and continuous over the whole of E, whose values are non-negative real numbers (actually numbers in the interval $0 \leqslant x \leqslant 1$), and such that $f(p)=0$, $f(q)=1$ if $q \epsilon F$. Also it is possible to construct ideal points by a process analogous to the definition of real numbers by classes of equivalent Cauchy sequences, and so to complete any space. The complete space is one in which every " Cauchy family converges " (pp. 17, 18) and in which the original space is everywhere dense.

Again, it is possible to define uniform continuity of a function whose argument and values belong to the same or different spaces with uniform structure. As with metric spaces a continuous function, defined over a compact space, is uniformly continuous (p. 24).

The book is clearly and attractively written and is an important contribution to axiomatic " point-set ", as apart from combinatorial, topology.

J. H. C. W.

Essai sur l'unité des Sciences mathématiques dans leur développement actuel. By A. Lautman. Pp. 60. 15 fr. 1938. Actualités scientifiques et industrielles, 589; le progrès de l'Esprit, IV. (Hermann, Paris)

Essai sur les notions de structure et d'existence en mathématiques. I. Les schémas de structure. Pp. 82. 20 fr. **II. Les schémas de genèse.** By A. Lautman. Pp. 83-162. 20 fr. 1938. Actualités scientifiques et industrielles, 590, 591; le progrès de l'Esprit, V, VI. (Hermann, Paris)

During some periods it seems as if there is a danger of mathematics breaking up into disconnected regions; salvation comes from unifying notions. Klein records that in a talk with W. K. Clifford both agreed that the task of connection had to be faced, and they suggested, as a test-case, projective geometry and number-theory. Readers of Klein's books will recall with what success he joined together the various branches of the mathematics of his time into a living unity. In those days group-theory was the great reconciler; to-day it is algebra and topology, the latter with one foot in point-set theory and the other in algebra.

It is one of the objects of the tracts under review to show the connections between various fields: formal logic, differential geometry, the Cartan-Weyl theory of continuous groups, uniformisation, class-fields, integral equations, the algebraic movement in analysis, the use of non-euclidean geometry in function-theory are all referred to, and the young mathematician will gain from the discussion an excellent idea of the present range of the science. A deeper purpose is a critical review of methods and of types of theorem in the various fields. There are two omissions: analytic number-theory as developed recently is not mentioned; this is perhaps not surprising, for the connections with the rest of mathematics are few and superficial, at present; but no mention is made of the Italian work on curves and surfaces, and this would have been a further illustration of the author's thesis. H. G. F.

Les Fondements expérimentaux de l'Analyse mathématique des faits statistiques. By G. Hostelet. Pp. 70. 15 fr. 1937. Actualités scientifiques et industrielles, 552; le progrès de l'Esprit, II. (Hermann, Paris)

Le Concours de l'Analyse mathématique à l'Analyse expérimentale des faits statistiques. By G. Hostelet. Pp. 70. 15 fr. 1938. Actualités scientifiques et industrielles, 585; le progrès de l'Esprit, III. (Hermann, Paris)

These tracts are concerned with methodology and the relation between the theory of probability and statistics on the one hand and experimental or observational data on the other. The views of K. Pearson, Fréchet and Reichenbach are examined, but the Russian school, the recent English workers and the movement to bring probability under the theory of measure are not considered. Those interested in the application of statistics to concrete instances will find here some useful criticism. H. G. F.

Leçons sur la théorie des spineurs. By E. Cartan. **I. Les spineurs de l'espace à trois dimensions.** Pp. 98. 25 fr. **II. Les spineurs de l'espace à n>3 dimensions: les spineurs en géométrie riemannienne.** Pp. 96. 25 fr. 1938. Actualités scientifiques et industrielles, 643, 701; exposés de géométrie, IX et XI. (Hermann, Paris)

In 1928 Dirac replaced the relativistic wave-equation of Quantum Mechanics, which in its simplest form is

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial z^2}+\frac{4\pi^2 m_0{}^2c^2}{h^2}\psi=0, \qquad \text{.................(1)}$$

by four first-order equations equivalent to *

$$\left.\begin{aligned}
\frac{h}{2\pi i}\left(\frac{\partial\psi_3}{\partial x}+i\frac{\partial\psi_3}{\partial y}+\frac{\partial\psi_4}{\partial z}-\frac{1}{c}\frac{\partial\psi_4}{\partial t}\right)+m_0c\psi_1=0,\\
\frac{h}{2\pi i}\left(-\frac{\partial\psi_4}{\partial x}+i\frac{\partial\psi_4}{\partial y}+\frac{\partial\psi_3}{\partial z}+\frac{1}{c}\frac{\partial\psi_3}{\partial t}\right)+m_0c\psi_2=0,\\
\frac{h}{2\pi i}\left(\frac{\partial\psi_1}{\partial x}-i\frac{\partial\psi_1}{\partial y}+\frac{\partial\psi_2}{\partial z}-\frac{1}{c}\frac{\partial\psi_2}{\partial t}\right)-m_0c\psi_3=0,\\
\frac{h}{2\pi i}\left(-\frac{\partial\psi_2}{\partial x}-i\frac{\partial\psi_2}{\partial y}+\frac{\partial\psi_1}{\partial z}+\frac{1}{c}\frac{\partial\psi_1}{\partial t}\right)-m_0c\psi_4=0.
\end{aligned}\right\} \qquad \text{.................(2)}$$

It is a consequence of these equations that each of the four functions ψ_1, ψ_2, ψ_3, ψ_4 satisfies the second-order equation (1), just as it is a consequence of the Maxwell-Lorentz equations in classical electromagnetism that each component of the electric and magnetic field-strengths satisfies the classical wave-equation.

Questions about the nature of equations (2) immediately arose. Did they conform to the fundamental principle of relativistic mechanics that all general physical laws are expressible in a form independent of the system of reference —a principle which, until then, had been understood to mean that all such laws could be written in tensor form? There was no difficulty about the original equation (1), which was simply the invariant equation

$$-\Delta_2\psi+\frac{4\pi^2 m_0{}^2c^2}{h^2}\psi=0 \qquad \text{..............................(3)}$$

for the space-time

$$ds^2=dx^2+dy^2+dz^2-c^2\,dt^2$$

* The equations given are not actually those of Dirac, but are certain linear combinations of them. They are based on equations (31) of p. 71, Vol. II, of the book under review. The ψ's are likewise linear combinations of Dirac's ψ's.

of Special Relativity, ψ being a scalar and $\Delta_2\psi$ Beltrami's second differential parameter for ψ. But there were serious difficulties in connection with equations (2). Since each of the four functions $\psi_1, \psi_2, \psi_3, \psi_4$ was known to satisfy (1), and hence (3), it was to be expected that they also would prove to be scalars for transformations of the space-time coordinates. Unfortunately they did not do so. As Dirac himself had shown, equations (2) and the functions $\psi_1, \psi_2, \psi_3, \psi_4$ display a kind of covariance when subjected to a Lorentz transformation, but their law of transformation, though linear, is not a tensor law in the ordinary sense. How, then, was Dirac's theory to be reconciled with the tensor principle of Relativity?

In 1929 van der Waerden showed that under a general Lorentz transformation the functions ψ_1 and ψ_2 transformed linearly and according to a definite law into a new pair $\psi_1{}', \psi_2{}'$, and that the functions ψ_3, ψ_4 likewise transformed linearly into a new pair $\psi_3{}', \psi_4{}'$. Thus the pair (ψ_1, ψ_2) and the pair (ψ_3, ψ_4) each displayed something of the characteristics of a vector of two components, though they could not be vectors in the ordinary sense because a vector in four-dimensional space-time has four components and transforms according to the tensor law. The pairs (ψ_1, ψ_2) and (ψ_3, ψ_4) were called *spin-vectors* or *spinors*, and their law of transformation a *spin-transformation*, the word " spin " being used because of their association with the theory of the spinning electron. So far as Special Relativity was concerned the principle of covariance was thus saved by admitting the upstart spinors and spin-transformations into the respectable company of proper tensors and tensor-transformations.

The extension of the theory to General Relativity proved more difficult. It could be done by introducing into space-time local Galilean systems of reference which had no physical significance in themselves, but it was felt by most mathematical physicists that theories depending upon the selection of such local systems were unacceptable. It was not long, however, before a different method was found. This depended essentially upon an analytical device which made spin-transformations independent of transformations of the space-time coordinates, a device adopted by Schouten, Veblen and other members of the Delft and Princeton schools of differential geometers, who attacked the problem geometrically and dealt not only with the extension of the theory of spinors to General Relativity but also with its extension to the Projective Relativity theories developed by those schools. In the projective theory of spinors the functions $(\psi_1, \psi_2, \psi_3, \psi_4)$ appeared as components of a single spinor, a distinction being made between the " four-component " and the " two-component " spinor—or, in the language now used by M. Cartan, between the spinor and the semi-spinor.

These developments occupied roughly the period from 1930 to 1934. Since then notable contributions have been made by Brauer and Weyl, who published an n-dimensional theory of spinors in 1935, and by E. T. Whittaker, who showed in 1937 how to find a set of proper tensor equations completely equivalent to those of Dirac.

What, then, *is* a spinor? It is easy to see in a general way that the two-component spinor of van der Waerden is, in fact, a geometrical entity which is familiar in a disguised form to every mathematical undergraduate. The equation

$$X^2 + Y^2 + Z^2 - c^2T^2 = 0$$

in Special Relativity is that of the null cone of vertex origin in the four-dimensional space-time ; or, if X, Y, Z, T be regarded momentarily as homogeneous coordinates in a three-dimensional projective space, it is the equation

of a quadric. The (imaginary) generators of this quadric are obtained in the usual way as the intersections of the pairs of planes

$$\left.\begin{array}{r}(X+iY)\lambda+(Z+cT)\mu=0,\\ -(X-iY)\mu+(Z-cT)\lambda=0\end{array}\right\} \qquad (4)$$

and

$$\left.\begin{array}{r}(X-iY)\lambda'+(Z+cT)\mu'=0,\\ -(X+iY)\mu'+(Z-cT)\lambda'=0,\end{array}\right\} \qquad (5)$$

λ/μ and λ'/μ' being the parameters that determine the individual generators. Regarded four-dimensionally these pairs of equations represent the generating planes of the null cone.

If X, Y, Z, cT are now respectively replaced by the operators

$$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z},\ -\frac{1}{c}\frac{\partial}{\partial t}\ \text{ and }\ \lambda,\ \mu,\ \lambda',\ \mu'\ \text{by}\ \psi_3,\ \psi_4,\ \psi_1,\ \psi_2,$$

equations (4) and (5) become precisely the differential terms of Dirac's equations (2).

Thus the theory of spinors is closely bound up with that of transformations of the generating planes of the null cone, a Lorentz transformation being one that leaves the null cone invariant. The kind of way in which the theory may be extended to n dimensions is now fairly obvious.

In the preface to the two volumes under review M. Cartan points out that, in their most general mathematical form, spinors were discovered by him in 1913 in his work on linear representations of simple groups, and he emphasises their connection, shown in Vol. II, with Clifford-Lipschitz hypercomplex numbers. In the text he develops the theory from the beginning, the first volume being devoted to spinors in three-dimensional space and the second to spinors in n-dimensional space, with special reference to the four-dimensional space-time of Special Relativity. The point of view is fundamentally geometrical, the theory being presented as one of linear representations of the rotation-groups of determinants $+1$ and -1. A spinor appears as " en quelque sorte un vecteur isotrope *orienté* ou *polarisé* "*: thus in a three-dimensional Euclidean space a spinor has two components (ξ_0, ξ_1) given in terms of the components (x_1, x_2, x_3) of a null vector (*i.e.* one for which $x_1^2+x_2^2+x_3^2=0$), by

$$\xi_0=\pm\sqrt{\left(\frac{x_1-ix_2}{2}\right)},\quad \xi_1=\frac{x_1+ix_2}{x_3}\xi_0\,;$$

if the plus sign is chosen, a rotation of the null vector through an angle 2π about the x_3-axis produces the minus sign, and *vice versa*, so that the choice of sign may be regarded as giving the spinor a polarisation which is altered by a rotation of the null vector.

In the last three chapters of the second volume the author gives special attention to the connections of spinor theory with Quantum Mechanics, the final chapter being devoted to the extension of the theory to Riemannian geometry. His theory of absolute differentiation depends, however, upon the selection of local Galilean systems of reference. He concludes by showing that, from the point of view adopted by himself, there can be no theory of covariant differentiation in the ordinary sense, a fact which accounts for the point of view of those writers who treat spin- and coordinate-transformations independently of one another. The latter point of view is described as " géométriquement et même physiquement si choquant "†, but it is probable that not everyone will agree with that description : the separation of the two kinds of transformation can, in fact, be given some geometrical plausibility, though

* Vol. I, pp. 52, 53.

† Vol. II, p. 91, footnote.

whether it is physically justifiable is more difficult to say. The truth of the matter is that ever since van der Waerden published his first paper on spinor analysis, practical quantumists have for the most part proceeded serenely on their way undisturbed by these mathematical anxieties. General spinor theories are of great mathematical interest and elegance and are necessary in order to reconcile the Dirac theory with the principle of covariance, but it can hardly be denied that they have so far cut but little physical ice.

M. Cartan's book will be indispensable to mathematicians interested in the geometrical and physical aspects of group theory, giving, as it does, a complete and authoritative survey of the algebraic theory of spinors treated from a geometrical point of view. It is not perhaps to be recommended as an introductory book, for it would be difficult to follow without some previous knowledge either of group theory in general or of spinor theory in particular, but it is and will remain a standard work. Whether it will prove to contain the last word on the subject of spinors remains to be seen. H. S. R.

Stellar Dynamics. By W. M. SMART. Pp. viii, 434. 30s. 1938. (Cambridge)

The rate of change of position of a star on the celestial sphere, expressed usually in seconds of arc per year in any suitable pair of angular coordinates (corrected, of course, for systematic effects like precession depending on the Earth's motion) is called its *proper motion*. It is more than two hundred years since Halley first realised that such motion exists, *i.e.* that the " fixed " stars are not really fixed but that over sufficiently long intervals of time changes in their relative positions can be detected. But when in 1783 Herschel published his first investigation of the Sun's motion relative to the neighbouring stars the actual value of the proper motion was known for only thirteen stars. Since then large numbers of proper motions of the brighter stars have been obtained from meridian-circle observations, but it was not until photographic methods were elaborated towards the end of the nineteenth century that proper motions of faint stars could be obtained. At the same time it became possible to measure the actual line-of-sight velocities of the stars from the Doppler shifts of their spectral lines. The great body of data which thus became available made possible early in the present century the pioneering work of Kapteyn, Eddington and Schwarzschild in the systematic study of stellar motions. This study has since been carried on by many astronomers and has recently produced such striking results as Oort's theory of the rotation of the galaxy. The leading British worker in this field at the present time is Professor Smart, the author of the book under review.

One of the first problems that arises is that tackled (though with such inadequate data) by Herschel—namely, to find the direction and magnitude of the Sun's velocity with respect to a mean standard of rest determined by the stars in its neighbourhood, say up to some hundreds of parsecs from it. It is clear in principle how the direction of the solar motion can be inferred from the proper motions of the stars, since the effect of the motion must be to cause the stars to appear statistically to open out in front of the Sun and to close in behind it. It is also evident that the stars in front must appear, statistically, to be approaching the Sun and those behind to be receding, thus making it possible to infer the magnitude of the solar motion from the radial velocities of the stars. Various mathematical techniques for handling these statistical effects are well established, and their application has yielded reliable knowledge of the solar motion.

It was at one time considered that, once the solar motion was known and allowed for, the *peculiar* motions of the stars would turn out to be quite hap-

hazard. However, Kapteyn's discovery of star streaming showed that this is not so. He found that actually the motions could be accounted for by supposing the stars to form *two* streams, containing comparable numbers of stars, moving in opposite directions in space, and such that in each stream the peculiar motions of the stars relative to its mean motion can be regarded as haphazard. Subsequent work has confirmed the fact that this gives a very satisfactory description of the observations. But it is not the only hypothesis to do so. About equally satisfactory is the ellipsoidal hypothesis of Schwarzchild, which postulates a velocity distribution function of the form

$$\exp\{-K^2U^2 - H^2(V^2 + W^2)\},$$

where U, V, W are the components of linear velocity of a star, U being measured in the direction of star streaming, and H, K are constants.

An important application of the foregoing work is the derivation of the mean distances of groups of stars from their proper motions. This again is simple in principle. Consider a group of stars which are known from their apparent luminosities to be all at approximately the same distance from the Sun. Suppose also that they are such that we can average out their peculiar motions, then the remaining systematic part of their proper motions represents the apparent motion resulting from the solar motion. Knowing the actual value of the latter, we can infer the mean distance of the stars concerned. In fact just as the orbital motion of the Earth makes possible the determination of trigonometric parallaxes of the nearer stars, so the solar motion makes possible the determination of parallaxes of more distant stars. But in the latter case the necessity of eliminating the peculiar motions, which are comparable with the parallactic motion, means that only *statistical* parallaxes are thus obtainable. Or, again, we may use the two-stream or ellipsoidal hypothesis to infer the mean transverse velocity of some class of stars, and by comparing this with their mean observed proper motion we can infer their mean distance. Clearly these mean distances can then be used to derive valuable information concerning the distribution of the stars in space.

At this stage one might argue that it should be unnecessary to *postulate* a velocity distribution function such as that given by the two-stream or ellipsoidal hypothesis. The observational data provide statistical information about the distribution of the stars in regard to apparent magnitude, proper motion, radial velocity, etc. Could we not then *deduce* the form of the unknown space-velocity distribution function? Actually we can, of course, write down relations between this unknown function and the observable statistical quantities. These relations take the form of integral equations whose formal solutions can in fact in most cases be obtained. But it appears that at present the data are not adequate to make this very general method of approach particularly fruitful in practice.

Up to this point the study is purely *kinematical*, but it is now possible to develop also the *dynamics* of a stellar system. The theory is closely analogous to the dynamical theory of gases, the stars now playing the part of the molecules in a gas. The only important differences arise from the circumstances that it can be proved legitimate to neglect " collisions " in any stellar system met with in practice, and that the relevant field of force is usually the gravitational field of the system itself. If the system is in a steady (or approximately steady) state, the dynamical equations provide general conditions which must be satisfied by the space-velocity distribution function.

Finally, all the main results obtained by applying the general theory to stars up to several hundred parsecs from the Sun, *i.e.* as far as stellar motions can at present be measured, find a rational explanation in the theory of galactic rotation. It is well known that the galactic system of stars forms a

flattened, roughly spheroidal, distribution, and according to this theory the individual stars describe orbits in the general gravitational field of the whole system. The majority of the orbits are nearly circular, and the mean velocity of the stars in any neighbourhood is a function of the distance from the centre of the galaxy. The variation with distance in the vicinity of the Sun is derivable from the observed distribution of proper motions and radial velocities, and permits an estimate of the distribution of mass in the galaxy. This and other features deduced from the theory agree well with the picture of the galactic system revealed by quite different methods.

Such in briefest outline (apart from a chapter on star clusters) is the field covered by Professor Smart's book. I have not summarised quantitative results, since the main emphasis appears to be on the general mathematical theory and on the technique of applying it to the data. The latter, as in all astronomical theory, is a very essential part of the work. It must be realised that all the observational effects are exceedingly small; the majority of known proper motions are only a few seconds of arc per century. Therefore all possible sources of error must be sought out and allowed for before the theory can be usefully applied. Professor Smart gives at all stages a full discussion of this aspect of the work, illustrating it by well-selected examples actually encountered in practice. He gives also the chief quantitative results of the work at each stage. Nevertheless, all this is done without in any way obscuring the main sequence of the theoretical development, which is handled in a masterly way from beginning to end.

Much of the mathematics involved is similar to that employed in the kinetic theory of gases. In other parts it is similar to that of general statistical theory and the theory of errors. As already indicated, the theory of integral equations is also introduced. The only criticism one would offer of this, or any other, aspect of the book is purely tentative; one suspects that a certain amount of the general theory could be set out more briefly, and the interpretation rendered more intuitive, were the notation of vector or tensor calculus adopted.

The book is evidently intended primarily for workers in this particular field. Anyone proposing to do research in it will find the book indispensable, for most of the work described has never before been brought together into a single connected text. And at the rate at which data are now accumulating it would be almost impossible for new workers to be recruited who could quickly make the best use of them without some such timely account of the present state of the subject.

However, the book can be read too with interest and profit by anyone desiring an introduction to this vastly important branch of modern astronomy. It makes no attempt to be popular and the writing, while always brisk, is never sensational. But it is extraordinarily refreshing to go through such a development of a relatively new subject which contains no conjectural steps and no noticeable logical gaps. One is convinced that any reader with fair mathematical equipment, but no specialised knowledge of the subject, will much more readily grasp its essentials from this book than he would from any effort that might be made to popularise it for him. W. H. McCrea.

Elementary Matrices and some Applications to Dynamics and Differential Equations. By R. A. Frazer, W. J. Duncan and A. R. Collar. Pp. xvi, 416. 30s. 1938. (Cambridge)

This book has been written with a very definite aim—to bring about a wider appreciation, among students in applied mathematics, of the conciseness and power of matrices and of their convenience in computation. All three authors

are primarily interested in Aeronautics and Aerodynamics, in which they are expert: Dr. Frazer and Mr. Collar collaborate in this work at the National Physical Laboratory while Professor Duncan occupies the Chair of Aeronautics at the University College of Hull.

The book presupposes no previous knowledge of matrices. Chapter I gives a very clear account of their fundamental properties and technical terms, emphasising rightly the importance of rectangular partitioning of larger into smaller matrices and bringing out clearly the significance of the product notation. For instance the expression $x'ax$ is used for the quadratic form $\sum_{i,j=1}^{n} x_i a_{ij} x_j$, where x' denotes a row-vector, a a symmetrical square array, and x the same vector arranged as a column. Chapter II develops the technique of matrix powering, and of differentiating and integrating matrices: here the row and column vectors of differential operators $\{\partial/\partial x_i\}$, $[\partial/\partial x_i]$ become significant. Chapter III gives a concise but clear account of canonical forms, quoting, with references, the difficult cases of non-linear elementary divisors and stating the results in an accessible form. Chapter IV begins the treatment which makes this book distinctive: it expounds a wide range of practical methods for handling and reducing matrices. The devices utilised are based on the experience over several years of recent work in the National Physical Laboratory and also in the Mathematical Laboratory of the University of Edinburgh, where theory and practice in determinant and matrix theory, on the part of both Professor E. T. Whittaker and Dr. A. C. Aitken, have been so happily blended. Calculation of complicated numerical determinants and their reciprocals, solution of numerical linear equations and the reduction of matrices to diagonal form are among the topics exemplified. The chapter ends with an application to the general algebraic equation in one unknown whose dominant root (if real) can be found by the iterative process invented by Daniel Bernoulli in 1728 and recently developed for the complex case by Aitken. Throughout this chapter careful attention is paid to the display of the calculations. They are extraordinarily like the systematic devices used by Pell and Wallis in their seventeenth-century volumes on algebraic equations. This chapter, and indeed the whole book, has been made possible through the impetus given to computational methods by the Edinburgh algebraic school and the pioneering work of *The Calculus of Observations* by Whittaker and Robinson.

Chapters V, VI and VII give a very detailed account of ordinary linear differential equations and the application of matrices to their solution, including the Peano-Baker iterative processes and the most recent work of Dinnik and Galerkin (1935), which appeared in a somewhat inaccessible Russian periodical on Aeronautical Engineering. Interesting tables are provided showing the relative accuracy of various approximative numerical processes, both for simple and for complicated examples.

Chapters VIII-XIII give a systematic account of kinematics, rigid dynamics, dynamical systems and the refinements in method due to imperfectly rigid bodies, the whole programme being designed to investigate the flight of an aeroplane. The classical equations of Lagrange and of Hamilton are worked out with clarity and brevity in matrix notation, which is particularly convincing in the passage on generalised components of momentum (p. 275). Forced vibrations, disturbed motions, torsional and flexural oscillations, the recently treated theory of solid friction and flutter problems are duly considered.

It is always gratifying to mathematicians when they find their abstract work acceptable and effective as a tool for a practical problem. For this reason the book will be of considerable interest to algebraists who, like the

present reviewer, have not yet made a detailed study of matrical applications. The present work deals with one of several such lines of enquiry: there are, for instance, equally valuable uses of the method in geometry and in statistics. The title of the book is well chosen in this respect, for it makes the scope of the subject matter completely clear.

The book will also be of interest to the applied mathematician and should convince him, if indeed such conviction is now necessary, of the breadth and power gained, once the inertia of learning a new notation and algorithm is overcome. It is a book which starts with the simplest beginnings and ends by going to the heart of a complicated problem. It is an excellent example of effective collaboration and it can be warmly recommended.

H. W. TURNBULL.

Introduction to the Calculus. Part I. By S. BEATTY and J. T. JENKINS. Pp. 647. English price 25s. 1938. (Toronto University Press; Humphrey Milford)

This is an elementary textbook concerned only with functions of a single variable: its scope is in some respects narrower than is usual in textbooks for the higher forms of English schools. The differential geometry of curves receives, for instance, scanty and merely incidental consideration but, on the other hand, there are academic discussions of the continuous functions, the definite integral of a continuous function and power series.

The authors clearly feel that the inordinate length of their work requires excuse. They say they " believe that real power in applying the Calculus and in thinking effectively about it must rest on a fairly complete appreciation of the ideas involved, as distinct from mere familiarity with the various technical rules applicable to normal cases ", and they add that " the characteristic feature of the present introduction is that explanations and illustrations are both numerous and varied and are couched in relatively simple terms ".

Had the concepts of the Calculus been lucidly explained and the examples and illustrations been apposite, prolixity in itself would not have been a sin. Indeed it might well be argued that mathematical textbooks suffer as a rule from undue compression. It is, however, open to doubt whether the objects aimed at have been attained. The worked examples, indeed, though somewhat elementary and occasionally pointless, as when the integral formula for arc is employed to find the length of a segment of straight line, are, on the whole, well conceived and interesting, but the explanations of the fundamental ideas are disappointing and fall far short of the standard of rigour usual in English textbooks with a similar purpose.

There is a tendency to regard the accumulation of numerical evidence as a valid method of procuring conviction of the truth of a formula. Thus when the rules for the differentiation of the trigonometrical functions have been stated but not proved, and a somewhat lengthy discussion has shown the rules to be not inconsistent with other known facts, the ratio $[f(x+h)-f(x)]/h$ is calculated for certain specified angles, tables to 15 places being used. We are then told that for two reasons, one of which is superfluous, nothing, strictly speaking, has been proved but " notwithstanding all this it is hoped that the student has now come to feel that the rules are true "; and finally, as an anticlimax, the fundamental result regarding Lt $\sin h/h$ is proved by showing that it is merely a case of Lt (chord/arc), which is axiomatically unity throughout the book.

The treatment of infinite series must also be mentioned.

On p. 207, there having been no discussion whatever of the problems of convergency, the series for $\sin x$, $\cos x$ are introduced and differentiated, the

sine and cosine of 3° are calculated both by trigonometric methods and by the use of the early terms of the series, and the resulting agreement between the values obtained is said (p. 210) to " imply a check on the correctness of the series for $\cos x$, $\sin x$ ".

The exponential e^x is defined as $10^{x \log_{10} e}$ and almost immediately identified, without proof, with the exponential series, which is then employed to confirm that $e^{x_1} \times e^{x_2} = e^{x_1 + x_2}$. Three " proofs " of the series for $\log(1+t)$ are given. Of the first, which depends upon the reversion of the exponential series, we are told that " we can never verify all the coefficients in any case. Nevertheless the method serves the purpose of leaving us almost entirely convinced that the logarithm formula is correct." We note that only four coefficients are calculated and that any conviction obtainable extends to the case where $|t| > 1$. The second equates coefficients of n in the two sides of

$$(1+t)^n = \exp\{n \log(1+t)\},$$

the result again being true apparently for all values of t, while the third is a mere term by term integration of the expansion of $(1+t)^{-1}$ with no discussion of the remainder terms.

Light relief is provided a little later when we are told that $\sqrt{-1}$ (not previously mentioned) is " donated by i " and that, by substituting the appropriate series, we can recognise at once that $e^{ix} = \cos x + i \sin x$, though it is rather a shock to learn that the discussion of the circular functions based on this formula serves as a model for the discussion of the hyperbolic functions.

The treatments of the continuous function, Taylor's series and the definite integral follow the accepted lines but more concise and clearer statements are available in well-known textbooks. The authors' ideal is evidently a minimum of symbolism : in pursuing it they achieve obscurity. There is much talk about what it is hoped to do, much afterwards about what is considered to have been done, but, at the crucial point, we are frequently met with " it is evident ".

Thus, discussing the Taylor expansion of e^x (p. 374), " it appears that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ "}$$

without further explanation and, when considering the definite integral, they treat as self-evident the continuity in x, for fixed h, of the oscillation of a continuous function in $(x, x+h)$.

The major defect of the book, to which its inordinate length is mainly due, is a lack of style both in the arrangement of material and the use of language. Many things might have been said differently and some were not worth saying at all. For instance a page of print (pp. 409-10) is consumed by the following passage in which the dots stand for words it is here unnecessary to repeat :

" A method of trial which might happen to supply an antiderivative with respect to x of $f(x)$ suggests itself. Take first a function $F(x)$ and compare $dF(x)/dx$ with $f(x)$. If these are the same then $F(x)$ is an antiderivative with respect to x of $f(x)$ and so all derivatives are of the form $F(x)+c$ and this means, therefore, that the first trial has been successful. If, on the other hand, these are not the same, take a second function $S(x)$ not of the form $F(x)+c$ and compare $dS(x)/dx$ with $f(x)$. If these are the same ... successful. If, on the other hand, $dS(x)/dx$ and $f(x)$ are not the same, take a third function $T(x)$ not of the form $F(x)+c$ and not of the form $S(x)+c$ and ... successful. If, on the other hand, $dT(x)/dx$ and $f(x)$ are not the same the succession of trials must be kept up. Continuing, therefore, the succession of trials, after the manner already indicated, it may happen that we discover eventually

(that is, in the final trial) an antiderivative $I(x)$. On the contrary, however, it may happen that we never discover an antiderivative $I(x)$ and this means, therefore, that the only information gleaned from the succession of trials is that none of the functions thus taken and tested is an antiderivative, and hence we are able to supply functions that are not antiderivatives but none that are. This latter does not imply any considerable advance towards the discovery of an antiderivative $I(x)$, since, after the first trial has been made, an infinite number of functions are left over untested and it is among these that all derivatives are found. It goes without saying that in testing functions to see if they are antiderivatives it is natural to test an apparently promising one before another seemingly less promising."

The final trial is evidently the Day of Judgment but is this a warning, a method or an expensive joke?

The book is clearly and attractively printed but much space has been wasted by inexperience in mathematical typesetting, or possibly by lack of an adequate fount. A much greater waste is due to the unnecessary " display " of formulae. On the first page $\sqrt{25-x^2}$ is displayed five times, three times alone and twice equated to y, while on pp. 67-69 there are twenty-six displays, $d^2y/dx^2=f''(x)$ accounting for ten of them. In these and other instances such prodigality might have been avoided by some slight resourcefulness in the use of language.

We regret this book and regret doubly its issue by a University Press.

J. L. B.

Introduction to the Theory of Equations. By Louis Weisner. Pp. ix, 188. 10s. 1938. (The Macmillan Company, New York)

This is an interesting and attractive book, compact and well written, with a distinct point of view and aim, namely by emphasis upon the concept of a field, to make of the theory of equations " a connected body of doctrine rather than a disjointed set of propositions ".

After an introductory chapter on complex numbers and the primitive roots of unity the author introduces the concepts of " number field ", " field of rational functions in a field " and " reducibility ", these leading up to a proof of the unique factorisation theorem. There follows a discussion of polynomials having assigned properties, such as the possession of given roots or the assumption of given values for given values of the variable. The Lagrange interpolation formula, with a useful extension, is given. The elementary method of finding the rational roots of an equation with integral coefficients leads naturally to Eisenstein's theorem on the irreducibility of certain polynomials. Following an admittedly somewhat slight treatment of continuity we pass to the problem of the isolation of real roots and the proofs of the theorems of Sturm, Budan and Descartes. The proofs of the two former seem excellent, the variations in sign of the relevant functions being clearly tabulated, but " Descartes " is, rather characteristically, deduced as a special case of " Budan ". Only two methods for approximating to roots, namely those due to Horner and Newton, are given, the circumstances in which the latter may be used being clearly stated. The succeeding chapter on resultants, discriminants and symmetric functions seems somewhat isolated and contains no mention of invariants.

A discussion of the algebraic extensions of a field, of conjugate and primitive elements and of radicals relative to a field leads on to the solution by radicals of the cubic and quartic equations. That for the cubic is supplemented by the usual trigonometrical treatment of the irreducible case but only one reduction of the quartic is given. The final chapters are devoted, the one to

the proof following Gordan of the fundamental theorem that every polynomial has a root, and the other to the possibility of constructions with ruler and compass. The latter concludes with Richmond's elegant construction of a regular polygon with seventeen sides.

It will be gathered that the treatment is more abstract and the discussion of certain topics thinner than would be usual in an introductory course in this country, where teachers prefer, as a rule, to journey from the particular to the general. Moreover, compression of the text has led to rather inadequate illustration of the fundamental concepts. These disadvantages are, however, much mitigated by sets of examples admirably designed to test comprehension of the theory and containing supplementary theorems which the student who avails himself of the hints given should have no difficulty in proving. There is also a good collection of miscellaneous examples. J. L. B.

Sane Arithmetic for Seniors. III. By C. Warrell. Pp. 64. Manilla, 1s.; limp cloth, 1s. 3d. 1939. (Harrap)

The third book of this arithmetic seems just as good as the other two, which have been commended in previous numbers of the *Gazette*. It contains thirty-six topics in everyday arithmetic, beginning with those depending on calculations with money—for instance, buying a cooking stove—and ending with those involving times, distances, speeds—for instance, travel from England to India by air. Amongst the topics included are graphs, life assurance, family catering. This book, though planned for senior schools, would infuse real life into the arithmetic lessons of any school containing children of 11+. C. T. D.

Trigonometry. By H. K. Hughes and G. T. Miller. With tables. Pp. viii, 189, 79. 7s. 6d. 1938. (John Wiley, New York; Chapman & Hall)

College Algebra. By L. J. Rouse. Pp. xiii, 462. 11s. 1939. (John Wiley, New York; Chapman & Hall)

Though intended primarily for American schools and colleges these two textbooks make sufficient contact with our own educational system. The *Trigonometry*, for instance, is definitely a first course in the subject, proceeding on orthodox lines as far as the solution of triangles and the compound angles. A refreshing novelty is the inclusion of a chapter on spherical trigonometry, described as introductory but actually containing enough to set the student of astronomy well on his way. A similar chapter is beginning to find a place in English schoolbooks and one hopes that the fashion will continue. Such a problem as, say, the calculation of lighting-up time appeals to most boys and may easily be the means of starting an abiding interest in mathematical astronomy.

To come back to the book under review, it has many commendable qualities; it is well produced, the text is clear and straight-forward, and the exercises instructive without being overwhelmingly numerous. Amongst useful features may be noted the insistence on the connection between Cartesian coordinates and trigonometrical functions, and the system of checks for the solution of triangles. An appendix, of some eighty pages, is devoted to tables, attractively arranged, and including one which will be found particularly helpful; this gives the circular measure as well as the natural functions for each ten minutes of arc.

In one or two matters of detail we are inclined to question the treatment. The book starts by discussing the standard placing of an angle—that is, with one arm along the x-axis of a coordinate system—and then proceeds, without further preamble, to define the various trigonometrical functions in terms of x, y and r. There is much to be said for this procedure, but our instinct, all

the same, is to let the necessity for the functions appear by taking a simple height and distance problem and showing how much better it can be solved by tables than by a scale drawing ; the treatment given in the text can follow in due course. Again, the formulae for sin $(x+y)$ and sin $(x-y)$ are deduced by methods apparently unrelated and the cosine formulae by taking complements of angles ; surely the ordinary projection method is preferable.

These are small criticisms and, taken as a whole, the book gives a clear and concise exposition of the subject matter and will well meet the requirements of teachers.

The *College Algebra* is a little harder to fit into the English scheme. It goes beyond School Certificate standard but is not sufficiently comprehensive for the mathematical specialist. There is a good chapter on the theory of equations, including Horner's method of solution, another on determinants, and the properties of complex numbers are worked out in some detail. There is, however, no discussion of convergency and the only series dealt with are the arithmetic, geometric and harmonic progressions. The binomial theorem, though proved for positive integral indices, is merely stated for other indices. As far as the treatment is concerned, the topics are well arranged, with clear and full explanations and a large number of worked examples. As in the *Trigonometry* reviewed above, the production leaves little to be desired. H. L.

Differential- und Integralrechnung. I, II, III. By O. Haupt, assisted by G. Aumann. Pp. 196, 168, 183. Geb. Rm. 11.20, 9.80, 10.60. 1938. Göschens Lehrbücherei, Reihe I, 24, 25, 26. (W. de Gruyter, Berlin)

This book, or set of books, falls partly into each of two classes. Its extreme generality and deep analysis place it in the class of fundamental treatises. On the other hand it gives weight here and there more particularly to less advanced aspects of the subjects treated and in other respects it tends more towards the textbook type. There are even occasionally a few examples set at the ends of chapters. There appears to be some transition occurring in the attitude in German universities towards mathematics. At the beginning, for example, it is stated that geometrical ideas are drawn upon to make a more intuitive (*anschaulich*) demonstration and it is true that some propositions, even in handling such general objects of thought as derived numbers of measurable functions, are expressed geometrically. The actual proofs, however, are all, very properly, arithmetical.

In the first three chapters there are clear signs of the teacher, and the account of convergent sequences is clear and very full. Very early (p. 38) there appears the Heine-Borel-Lebesgue covering theorem.

The first volume is about sequences, sets of points and functions of a real variable or combination of variables. There is a long discussion of monotone and convex functions. The second volume concerns itself with differentiation and there is a brief aside on the inverse operation. This differentiation is made to depend on the range D of f—that is, on the set of points where $f(P)$ is defined. For example, in two dimensions a distinction is stressed in the book between " differentiability " and " weak differentiability ". This distinction is only important when D, to speak loosely, is in a sense one-dimensional near P_0. The third volume handles integration and follows a part of the procedure of Carathéodory's *Reelle Funktionen*, but although it includes the Lebesgue integral a great many pages are devoted to Jordan content and the Riemann integral.

In the first volume, but not in the other two, there is some laxity of expression and on page 75 there is a mistake which is rather common. It is stated that if at x_0, $f(x)$ possesses a proper or improper (*i.e.* ∞) limiting value y_0 ; if

at y_0, $F(y)$ possesses a limiting value, and if the range of values of $f(x)$ is in the range of y for which $F(y)$ is defined, then $F[f(x)]$ possesses a limiting value x_0. A counter-example is

$$x_0=y_0=0,\ \ f(x)=x\sin 1/x,\ \ x\neq 0,\ \ F(y)=1,\ \ 0<|y|\leqslant 1,\ \ F(0)=0.$$

Other well-known German books give a specific warning against this mistake.

The book will no doubt be found very useful in Germany. For English students it has less importance. Its generality makes it unsuitable to serve the purposes of the many preparatory English books. On the other hand, for the expert there are a number of advanced treatises, in English, French and German, which are more complete on each particular topic. P. J. D.

Functional Arithmetic through Citizenship. III. Ways and Means. By J. Trevelyan and J. Morley. Pp. vi, 90. 1s. 6d. 1939. (Longmans)

In an earlier review of Books I and II (*Gazette*, May, pp. 231-2) I discussed the principles and aims of the series of which this is Book III. This book covers the following topics: work and unemployment, insurance and pensions, savings and banks, hire-purchase, gas and electricity. The main concern of the book is the teaching of percentage.

Briefly the aims of the books are:

(1) to relate what is taught in school to life outside school;
(2) to teach processes of arithmetic through aspects of everyday life;
(3) to give training in serious reading rather than in " doing sums ".

(1) In Books I and II arithmetic was related to the needs of a person as part of a family. In this book the fundamental idea is membership of a community. Underlying the subject matter and teaching of this book is the idea that each person gains immense advantage through this membership, especially in increased security against accident and misfortune. It will probably seem rather unusual for an arithmetic book to have a message, but this point, though never actually stated, seems to me exceedingly well made.

(2) This aim is carried out. Sometimes in Books I and II there was a feeling that the connection between the subject matter and the arithmetic was forced: in this book there is a real and natural unity between the two. The arithmetic is part of the explanation, for example, in making clear both aspects of percentage. In considering unemployment in different industries, there are 122,625 miners unemployed out of 972,680, and 36,290 shipbuilders out of 172,810. How can we compare these? The actual arithmetic is not done at first but the numbers of unemployed per 100 workers is shown. It is stressed that 100 is the unit that we all agree to take and therefore there is value in a symbol for " per 100 ". Similarly a rate of $x\%$ arises naturally from insurance and saving as an economy of writing and explanation.

(3) As serious reading the book seems to me really excellent. The explanations do not lose themselves in detail yet there is a correct idea of the principles involved; the illustrative tables are valuable in training a child to appreciate the significance of figures; the sketches are amusing and full of point; there is constant help and encouragement given to the reader to collect similar information for himself.

The book would seem to be most useful for the second year of a secondary school or the equivalent in a senior or central school. The understanding of percentage (which the frequency of the question " Do we put the 100 underneath or on top? " shows is often lacking) is the keystone of arithmetic and the book gives a slow and concrete introduction to this.

The series is exerting a real fascination on me and I am looking forward to receiving Book IV. D. E. S.

La mécanique ondulatoire des systèmes de corpuscules. By L. de Broglie. Pp. vi, 223. 100 fr. 1939. Collection de Physique mathématique, 5. (Gauthier-Villars)

This book is a sequel to others by the author which deal with the wave-mechanics of a single particle in a given field of force. It is concerned with the extension of the theory to systems of particles, and in particular with those parts of the theory which have been rather neglected by other writers.

It is interesting to compare the subjects treated in books on classical dynamics and on quantum mechanics. In the ordinary textbooks on dynamics, considerable attention is paid to general theorems concerning the integrals of the equations of motion and to the separation of the motion of the centroid from the motion relative to the centroid. In books on quantum mechanics, on the other hand, the n-body problem holds the field, because nature has set the mathematical physicist questions which he must solve, or else he must cease to work, and so approximate methods have been evolved for dealing with the n-body problem, which in the classical theory has proved so intractable. The analogues of the general theorems of classical mechanics are hardly ever touched upon; even when the angular momentum is discussed, it is the property that the angular momentum must be quantised which is stressed and this has no analogy in classical mechanics. A large part of M. de Broglie's book is devoted to the discussion of these neglected general theorems.

The book may be roughly divided into two parts. The first deals with the general theory of the first integrals of quantal systems; the theory of the motion of the centroid has a whole chapter to itself. The second part is mainly concerned with the general symmetry properties of energy levels of complex systems containing a number of identical particles. The discussion is on an elementary level throughout, complicated methods such as that of Slater being avoided and the only special problems treated being the helium atom and the hydrogen molecule.

The exposition is clear and easy to follow and the book can be recommended as an introduction to the more advanced treatises. A. H. W.

An Introduction to Symbolic Logic. By S. K. Langer. Pp. 363. 12s. 6d. 1937. (Allen & Unwin)

It is now well known that the application of mathematical methods to logic has produced revolutionary advances in that subject during the last hundred years, comparable in theoretical importance to those which have simultaneously occurred in physics and mathematics. Hitherto there has been no good introductory textbook to which beginners might be referred. Russell's in many ways excellent *Introduction to Mathematical Philosophy* proves too difficult for all except the most persistent and has rather the purpose of making *Principia Mathematica* and its doctrines more intelligible than that of easing the study of symbolic logic. The same might be said of Carnap's *Abriss der Logistik*, which has the extra disadvantages of being written in German and not easily obtainable in England. Even such a satisfactory textbook as Stebbing's *Modern Introduction to Logic* is largely concerned with the traditional problems of Aristotelian logic and is unable to devote sufficient space to the new, generalised logic and its technique.

Here at last is a book, written clearly and in a simple style, which one can hand to beginners with every confidence that their curiosity will not be stifled by over-elaboration of doctrine. Dr. Langer's experience of teaching symbolic logic has prompted her wisely to use as few symbols as is compatible with clarity and to prepare the introduction of each abstract notion by familiar illustrations drawn from everyday life. In this way the reader finds himself

carried painlessly from general discussions on such topics as Form, Structure, Abstraction, Symbolism, Formal Context and Systems to chapters in which he is introduced to Boolean Algebra and the Calculus of Propositions. That each chapter is commendably brief and concludes with "Questions for review" and "Suggestions for class work" is further evidence that the book has been written by one who has constantly regarded the interests of the learner. (The expert will be interested to notice the emphasis on Boole's mode of presenting the Algebra of Logic and the influence of the work of Sheffer and Huntington.)

A book whose clarity and skill of presentation deserve high praise. M. B.

Grundzüge der Theoretischen Logik. By D. Hilbert and W. Ackermann. 2nd edition. Pp. viii, 133. RM. 9.60; geb. RM. 10.80. 1938. Grundlehren der mathematischen Wissenschaften, 27. (Springer, Berlin)

This is the second edition of the well-known introduction to symbolic logic which first appeared ten years ago. Few changes have been made in the treatment, which is still conspicuous for rigour and clarity. The first two chapters, on the propositional calculus and the calculus of classes respectively, are substantially unaltered. The chapter on the restricted calculus of propositional functions has been fairly extensively revised and contains new proofs of the independence and categoricity of the axioms used. Omission of the so-called branching theory of types (*verzweigte Typentheorie*), which subsequent criticism has shown to be an unnecessary complication of Russell's original Theory of Types, has allowed the discussion of the general calculus of propositional functions (*erweiterte Prädikatenkalkül*) to be improved and condensed. Slight changes of terminology have been made throughout to bring the work into line with Hilbert and Bernays' *Grundlagen der Mathematik* and occasional references to recent researches have been inserted.

The changes made have increased the value of what was already a first-rate book indispensable to the serious student of mathematical logic. M. B.

Advanced Calculus. By W. B. Fite. Pp. xii, 399. 21s. 1938. (Macmillan, New York)

The preface says that this book is "written to supply an introductory course in mathematical analysis for those who are looking forward to specialising in mathematics". It gives an account of the elementary part of real variable theory, including a chapter on Fourier series, and concludes with two additional chapters, one on Calculus of Variations and the other an introduction to complex variable theory.

The author has given a generally very clear account of his material. He has not attempted to prove results in the most general form where this would complicate the argument, but has given them under carefully stated conditions sufficiently general for the applications in view. It is difficult to preserve a strictly logical order and give interesting applications early in the subject. Giving applications early has enabled the author to show the necessity of considering later theory. The cost has been many forward references (perhaps not so many as is the reviewer's impression, because they are liable to become irritating, especially in the form of one in § 91 where we read "but we know from § 102 that . . .").

There are just a few points which seemed a little obscure. The main ones are the definition of limit *, the proof that $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$ (probably due to

* The term "upper limit" is used to mean "upper bound", "variable" sometimes means "function", and $\overline{\lim}\, a_n$ is called "the greatest limit" of the sequence a_n.

trying to over-simplify the account of the case in which $u - u_0$ has zeros in the neighbourhood of the point considered), and the account of differentials and partial derivatives. And why was the use of differentials *half* abandoned in the discussion of change of variables?

Taylor's expansion and Fourier series are each nicely introduced by a brief reference to the idea of approximating to a given function by functions of a simple selected type.

The book contains a good selection of examples. R. C.

The Theory of Functions. By E. C. Titchmarsh. Second edition. Pp. x, 454. 25s. 1939. (Oxford)

This second edition has been printed photographically from corrected sheets of the first. Some slight rearrangements have made possible the insertion of pages 284a-284g, containing an introduction to the study of asymptotic values and meromorphic functions, which will pave the way for the student's reading of Nevanlinna's books in the *Collection Borel* and in the Springer *Grundlehren* series.

So many fields of work are mentioned in this volume that it is almost surprising to note how far and how thoroughly each is discussed ; the explanation lies, of course, in the author's effortless mastery of technique. His style is a sensible compromise between the " no words " and the " no symbols " schools.

It is enough, in these days, to refer to " Titchmarsh ", evidence that it is now a classic. But lest anyone reading this notice should not know the book, it may be added that it is the best English one-volume introduction to the modern theory of functions of real and complex variables. T. A. A. B.

CORRESPONDENCE.

REFERENCE FOR SIMILAR TRIANGLES

To the Editor of the *Mathematical Gazette*

Sir,—In a letter in the May *Gazette* Mr. Inman stated that in Germany the symbols used for similarity and congruency are respectively $\sim$ and $\simeq$. The same use of these symbols obtains in the United States. I see no reason why we should not adopt the symbol $\sim$ for similarity in England ; there is very little likelihood of any confusion of meaning arising on account of our present usage of the symbol in other connections. Yours truly, F. J. Wood.

To the Editor of the *Mathematical Gazette*.

Dear Sir,—The recent correspondence in the *Gazette* on Signs Suitable for Similarity tempts me to expose an idea on which I have been ruminating since the matter was first broached. It seems to me that the " squiggly " suggestions put forward are likely to prove somewhat awkward to write (to say nothing of subsequent reading!) and I propose a symbol as closely as possible resembling the now well-established " $\equiv$ " for congruence, viz., " $|||$ ".

Such a sign would have the advantages : (*a*) of ease of both writing and reading ; (*b*) of a resemblance to the congruence symbol comparable with the resemblance between the cases of congruence and those of similarity ; and (*c*) an association with the idea of parallelism (symbolised by " $||$ ") which is so often—though admittedly not always—bound up with similar triangles. Thus we should write (*e.g.*) :

For congruence—$\triangle ABC \equiv \triangle XYZ$ (SAS).

For similarity —$\triangle ABC \,|||\, \triangle XYZ$ (SAS).

I am, G. H. Grattan-Guinness.

MATHEMATICAL FILM

The following review of a mathematical film is reproduced from the *Monthly Film Bulletin* of the British Film Institute by permission of the Institute.

Hypocyclic motion, A. (Great Britain) 1938. 35 mm. Silent (flam.). 16 mm. Silent (non-flam.). 9·5 mm. Silent (non-flam.). 750 ft., 12 min. 1 reel when hired, 3 reels when purchased.

Production : Produced by R. A. Fairthorne and B. G. D. Salt.

Description : Cine-diagrams.

Purpose : Aiding the teaching of mathematics and kinematics.

Teaching Notes : Available.

Distributors : B. G. D. Salt, 5 Carlingford Road, Hampstead, London, N.W. 3; and National Film Library, British Film Institute, 4 Great Russell Street, London, W.C. 1.

Conditions of Supply : This film, which should be purchased if it is to be of teaching value, is intended to be used in short lengths to illustrate one point at a time. For this reason it is not sold as one complete reel. The only value in hiring the film is for general demonstration purposes, and for this reason it is supplied as one complete reel for hire. Apply to distributors.

Contents : The film consists of a series of moving diagrams based on the geometry of hypocyclic motions. In the first reel a rigid bar moves with its ends on two fixed straight lines. The motion of the instantaneous centre relative to the fixed and the moving planes is clearly shown, and the motion is demonstrated to be equivalent to that produced by the rolling of one circle upon another. In this reel, as in the remainder, the distinction between points and lines fixed in space or rigidly connected with the moving plane is well shown by the use of black and white lines respectively.

In Reel 2 points on the circumference of the rolling circle trace out diameters of the fixed circle, and the motion of a point on the moving plane is shown to be capable of generation by another epicyclic motion.

In Reel 3 the demonstration is extended to other points in the plane of the moving circle. An ellipse is generated both as a locus and as an envelope of a carried line. In an interesting sequence the generation of the ellipse is connected with the use of the elliptic chuck.

Appraisal : The description of the contents may give a misleading impression of the complexity of this sequence of films. In fact the merits of these reels can only be appreciated by seeing them. They succeed in presenting somewhat complex relations in a form which is both intelligible and memorable. All the dominant relations involved in this important type of motion are clearly and vividly demonstrated. The mathematician, moreover, will not be surprised to find himself deriving much incidental aesthetic satisfaction from watching the film. Any teacher who wishes to apply " Stage A " or intuitive methods in kinematics or geometry should find these reels of great value; they can hardly fail to strengthen the intuitive grasp of geometric and dynamic relationships which is of such fundamental importance in the learning of mathematics. It deserves to be mentioned that the three reels are not intended to be shown consecutively (though they may be connected for purposes of revision), and they are not accompanied by explanatory captions. This is a virtue which makers of other films might well imitate, since it leaves the teacher free to provide his own explanation and to fit the moving diagrams in the most convenient manner into his own scheme of work.

Suitability : Pre-School Certificate as an illustration of loci; first-year sixth form; and subsequently for the beginning of kinematics.

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